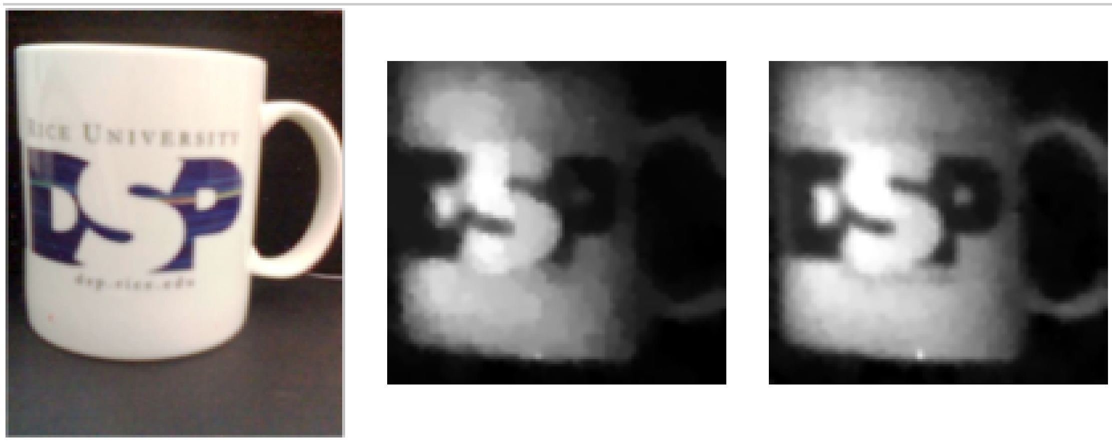
COMPRESSIVE SAMPLING

[A sensing/sampling paradigm that goes against the common knowledge in data acquisition]

[Emmanuel J. Candès and Michael B. Wakin]



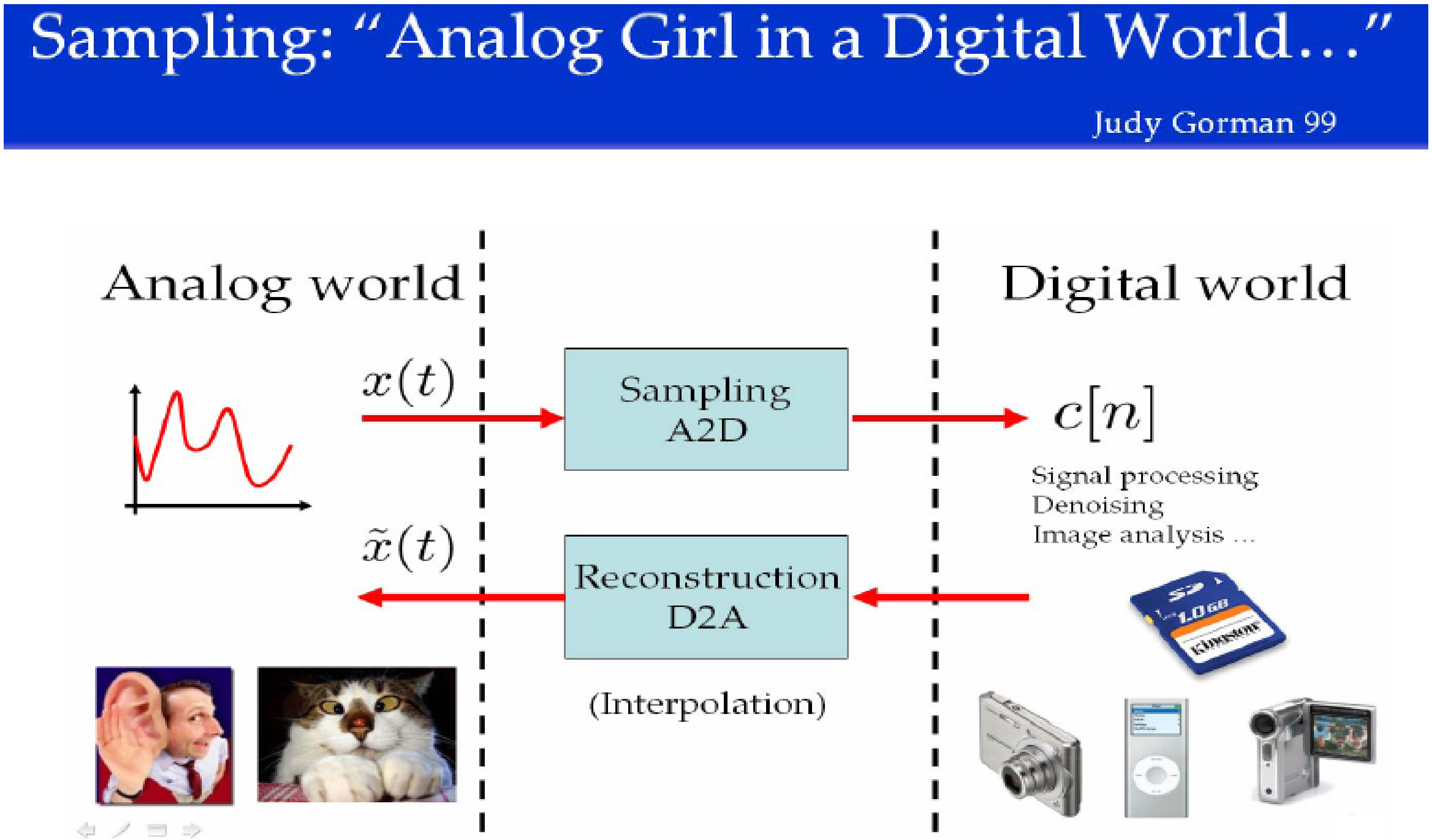
Original Object	4096 Pixels 800 Measurements (20%)	4096 Pixels 1600 Measurements (40%)

ENDRA

INTRODUCTION

Digital Revolution

INTRODUCTION

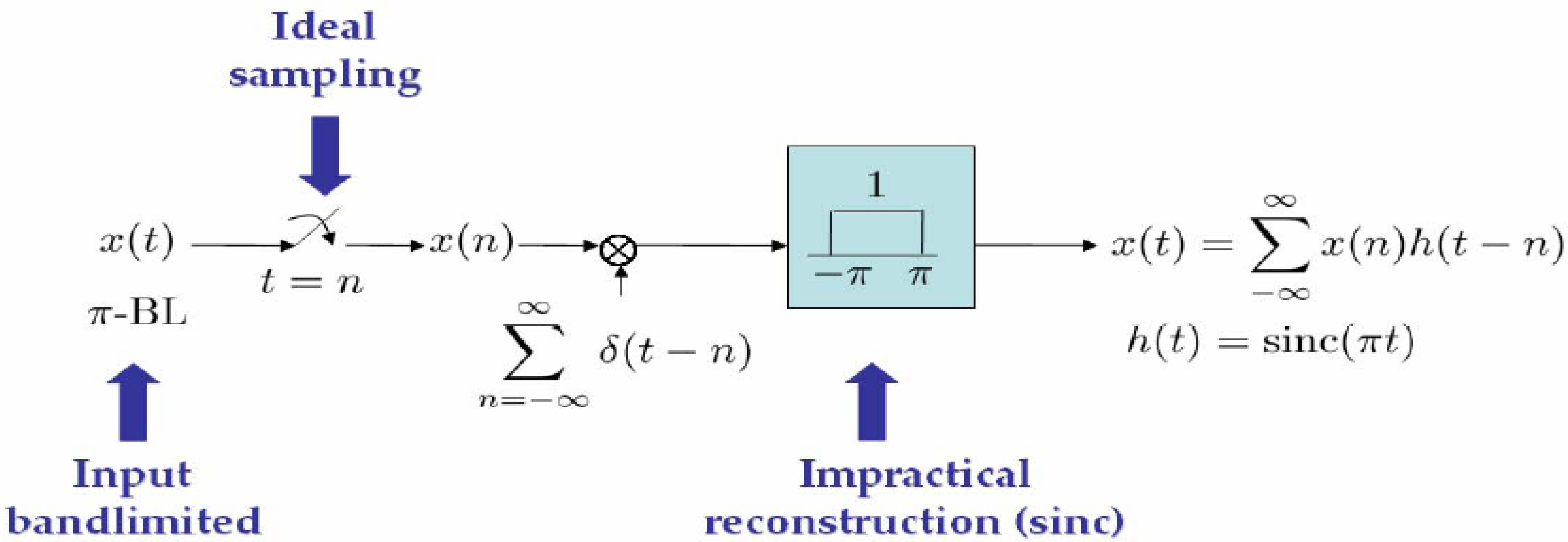


DSP revolution: sample then process

Data Acquisition

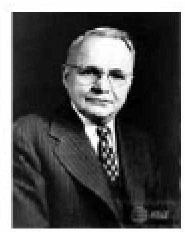
Shannon-Nyquist Sampling Theorem :

"No information loss if we sample at 2x the bandwidth"



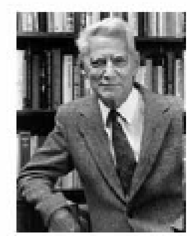
"Success has many fathers ..."

Whittaker
1915



Nyquist
1928

Kotelnikov
1933



Shannon
1949

Data Acquisition

Pressure is on Digital Sensors

- Success of digital data acquisition is placing increasing pressure on signal/image processing hardware and software to support

higher resolution / denser sampling

» ADCs, cameras, imaging systems, microarrays, …

x

large numbers of sensors

» image data bases, camera arrays, distributed wireless sensor networks, …

x

increasing numbers of modalities

» acoustic, RF, visual, IR, UV

=

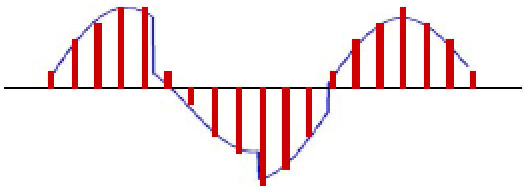
deluge of data

» how to acquire, store, fuse, process efficiently?

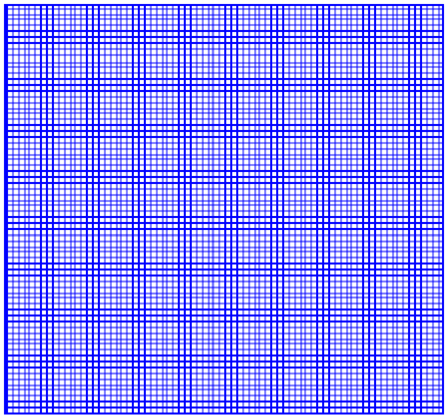
Data Acquisition

Trends (demands):

- faster sampling
- larger dynamic range
- higher-dimensional data
- lower energy consumption
- new sensing modalities



time



space

Old-fashioned Thinking

Long-established paradigm for digital data acquisition

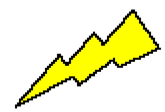
- uniformly ***sample*** data at Nyquist rate (2x Fourier bandwidth)
- ***compress*** data

$N \gg K$

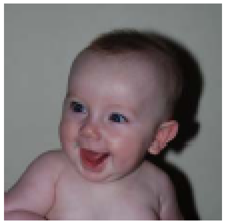
$x \rightarrow$ [sample] $\xrightarrow{N}$ [**compress**] $\xrightarrow{K}$ [transmit/store]

Shannon sampling theorem

JPEG
JPEG2000
...



[receive] $\xrightarrow{K}$ [decompress] $\xrightarrow{N}$ $\widehat{x}$



Old-fashioned Thinking

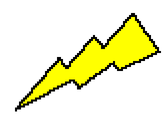
What's Wrong with this Picture?

Why go to all the work to acquire** N* ***samples only to discard all but** K* ***pieces of data?

$$N \gg K$$

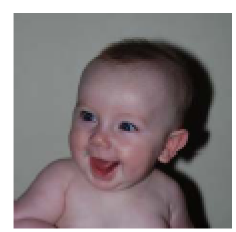
$$x \longrightarrow \boxed{\text{sample}} \xrightarrow{N} \boxed{\textbf{compress}} \xrightarrow{K} \boxed{\text{transmit/store}}$$

JPEG
JPEG2000
...



Shannon sampling theorem

$$\boxed{\text{receive}} \xrightarrow{K} \boxed{\text{decompress}} \xrightarrow{N} \widehat{x}$$



Old-fashioned Thinking

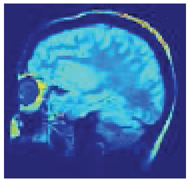
- Waste the imaging time, imaging sensors, and power!
- Especially for on-board cameras in satellites for large-scale remote sensing.

The same problem for…

- Remote sensing

- Communication
- ……

- CT, MRI



COMPRESSIVE SAMPLING
Makes Every Pixel Count

> CS THEORY ASSERTS THAT ONE CAN RECOVER CERTAIN SIGNALS AND IMAGES FROM FAR FEWER SAMPLES OR MEASUREMENTS THAN TRADITIONAL METHODS USE.

A precursor of compressed sensing was seen in the 1970s, when seismologists constructed images of reflective layers within the earth based on data that did not seem to satisfy the Nyquist-Shannon criterion : “Sparse Spike Train Hypothesis”.

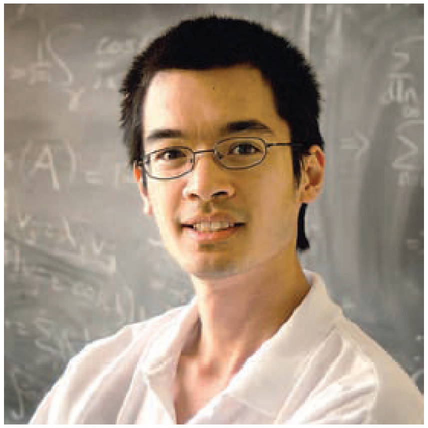
The ideas behind compressive sensing came together in 2004 when Emmanuel J. Candès, a mathematician at Caltech, was working on a problem in magnetic resonance imaging. He discovered that a test image could be reconstructed exactly even with data deemed insufficient by the Nyquist-Shannon criterion.

COMPRESSIVE SAMPLING
Makes Every Pixel Count

Emmanuel Candes of Caltech

People involved

Terence Tao of the University of California



Justin Romberg of Georgia Tech

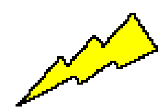
David Donoho of Stanford University

COMPRESSIVE SAMPLING
Makes Every Pixel Count

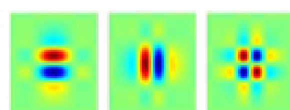
- Directly acquire "*compressed*" data
- Replace samples by more general "measurements"

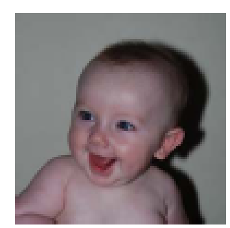
$$K \approx \underline{M} \ll N$$

$x \rightarrow$ **compressive sensing** $\xrightarrow[y]{M}$ transmit/store



receive $\xrightarrow{M}$ reconstruct $\xrightarrow{N}$ $\widehat{x}$





COMPRESSIVE SAMPLING
Makes Every Pixel Count

CS relies on two principles:
Sparsity, **which pertains to the signals of interest, and**
Incoherence, **which pertains to the sensing modality.**

Candès and Tao argue that compressive sensing is based on a kind of uncertainty principle, where the spectrum of the signal and that of the measuring instrument have complementary roles.

The traditional sensing strategy takes sharply focused samples; it's like hunting with a spear. This works well when the signal is spread out over a broad domain.

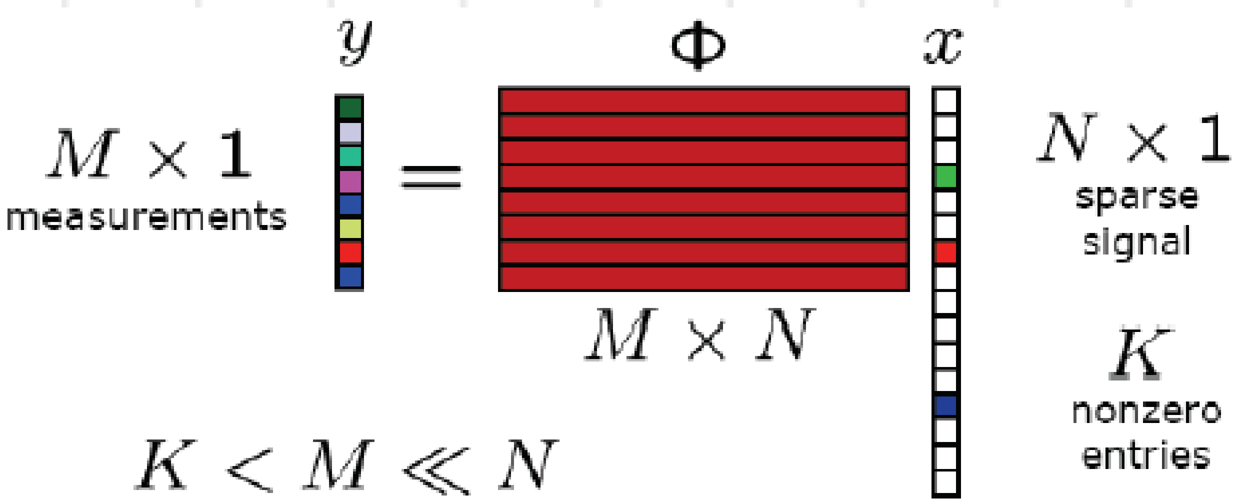
But when the signal itself is highly structured and narrowly focused, the better plan is to spread the measurements out over the domain—to hunt with a net rather than a spear.

COMPRESSIVE SAMPLING

Technical

Mathematically

$$y = \Phi x + \epsilon. \qquad \Phi \in C^{K,N},\ K << N$$



Hopeless to solve?

COMPRESSIVE SAMPLING
Technical

Fundamental problem

- Finding sparse solutions to the underdetermined inverse problems.

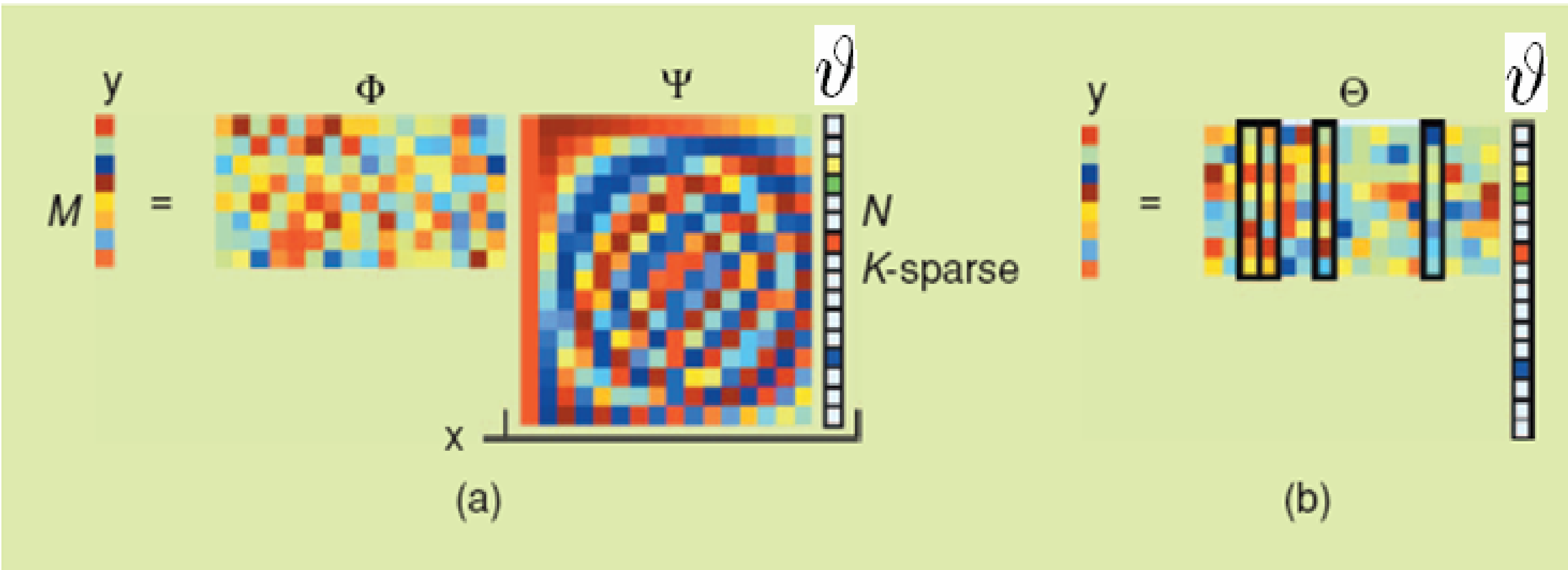
- Recover the original data from a few linear measurements.

COMPRESSIVE SAMPLING

Technical

If x is compressible

$$y = \Phi\Psi\vartheta + \epsilon.$$



COMPRESSIVE SAMPLING

Technical

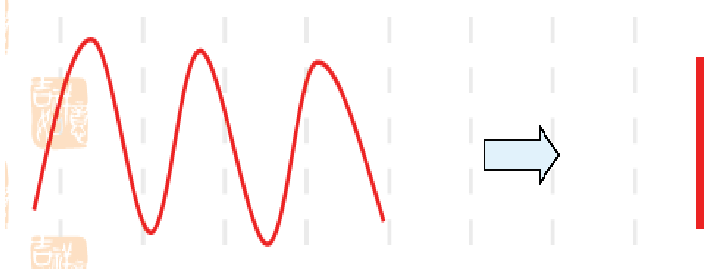
CS theory requires three aspects

1. The desired signals/images are compressible
2. CS matrix satisfies RIP (restricted isometry property): noise-like incoherent
3. Nonlinear recovery algorithm

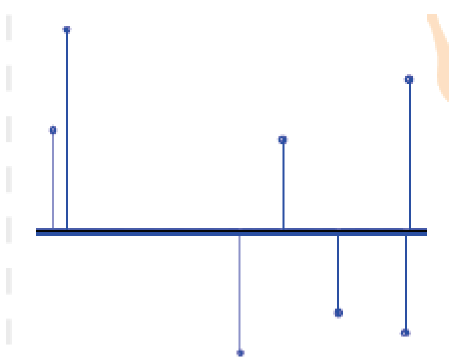
1. Compressibility

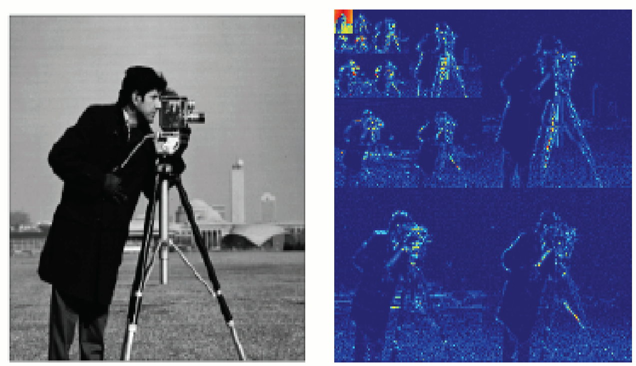
MANY NATURAL SIGNALS ARE SPARSE OR COMPRESSIBLE IN THE SENSE THAT THEY HAVE CONCISE REPRESENTATIONS WHEN EXPRESSED IN THE PROPER BASIS.

- Transform sparse



- spatial sparse





2. RIP condition

- What are good CS matrices? how many measurements are required for successful reconstruction?
- RIP: a sufficient condition (E. Candes, T. Tao)

Let A be an $m \times p$ matrix and let $s < p$ be an integer. Suppose that there exists a constant δ_S such that, for every $m \times s$ submatrix A_s of A and for every vector y,

$$(1-\delta_s)\|y\|_{\ell_2}^2 \le \|A_s y\|_{\ell_2}^2 \le (1+\delta_s)\|y\|_{\ell_2}^2.$$

Then, the matrix A is said to satisfy the s-restricted isometry property with restricted isometry constant δ_S.

In linear algebra, the restricted isometry property characterizes matrices which are nearly orthonormal, at least when operating on sparse vectors. The concept was introduced by Candes and Tao[1] and is used to prove many theorems in the field of compressed sensing.[2]

2. RIP condition

How to construct the CS matrices

- A basic rule: CS matrix should be incoherent in the sparse transform domain.

The coherence between the sensing basis Φ and the representation basis Ψ is

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot \max_{1 \le k, j \le n} |\langle \varphi_k, \psi_j \rangle|.$$

$$\mu(\Phi, \Psi) \in [1, \sqrt{n}]$$

2. RIP condition

How to construct the CS matrices

- The greater incoherence of the measurement / sparsity pair (Φ, Ψ), the small measurements needed.

Fix $f \in \mathbb{R}^n$ and suppose that the coefficient sequence x of f in the basis Ψ is S-sparse. Select m measurements in the Φ domain uniformly at random. Then if

$$m \geq C \cdot \mu^2(\Phi, \Psi) \cdot S \cdot \log n$$

One suffers no information loss by measuring just about any set of m coefficients which may be far less than the signal size apparently demands. If $\mu(\Phi, \Psi)$ is equal or close to one, then on the order of $S \log n$ samples suffice instead of n.

2. RIP condition

How to construct the CS matrices

The signal f can be exactly recovered from our condensed data set by minimizing a convex functional which does not assume any knowledge about the number of nonzero coordinates of x, their locations, or their amplitudes which we assume are all completely unknown a priori. We just run the algorithm and if the signal happens to be sufficiently sparse, exact recovery occurs.

- Random matrices: Gaussian, random partial orthogonal matrices.

- How to construct deterministic and explicit CS matrices is open? [DeVore,Indyk]

3. Recovery algorithms

There are two feasible ways to recover the data :

- *Matching pursuit*: locate a wavelet whose signature seems to correlate with the data collected; remove all traces of that signature from the data; and repeat until we have totally "explained" the data collected in terms of wavelet signatures.

- *Basis pursuit* (or l^1 minimisation): Out of all the possible combinations of wavelets which would fit the data collected, find the one which is "sparsest" in the sense that the total sum of the magnitudes of all the coefficients is as small as possible. (It turns out that this particular minimisation tends to force most of the coefficients to vanish.) This type of minimisation can be computed in reasonable time via convex optimisation methods such as the simplex method.

3. Recovery algorithms

- LP (linear programming), reweighted LP [Candes et al.]

Finding the candidate with the smallest *L_1* norm can be expressed relatively easily as a linear program, for which efficient solution methods already exist.

- **Minimum ℓ_1 norm reconstruction:** Surprisingly, optimization based on the ℓ_1 norm

$$\widehat{s} = \operatorname{argmin} \|s'\|_1 \quad \text{such that} \quad \Theta s' = y$$

3. Recovery algorithms

- Greedy/OMP (orthogonal matching pursuit) StOMP(stagewise OMP) [Tropp, Donoho, et al.]
- GRSP(gradient projection sparse reconstruction) [Figueiredo et al.]
- Iterative thresholding [Daubechies, Starck, et al]

3. Recovery algorithms

Iterative curvelet thresholding

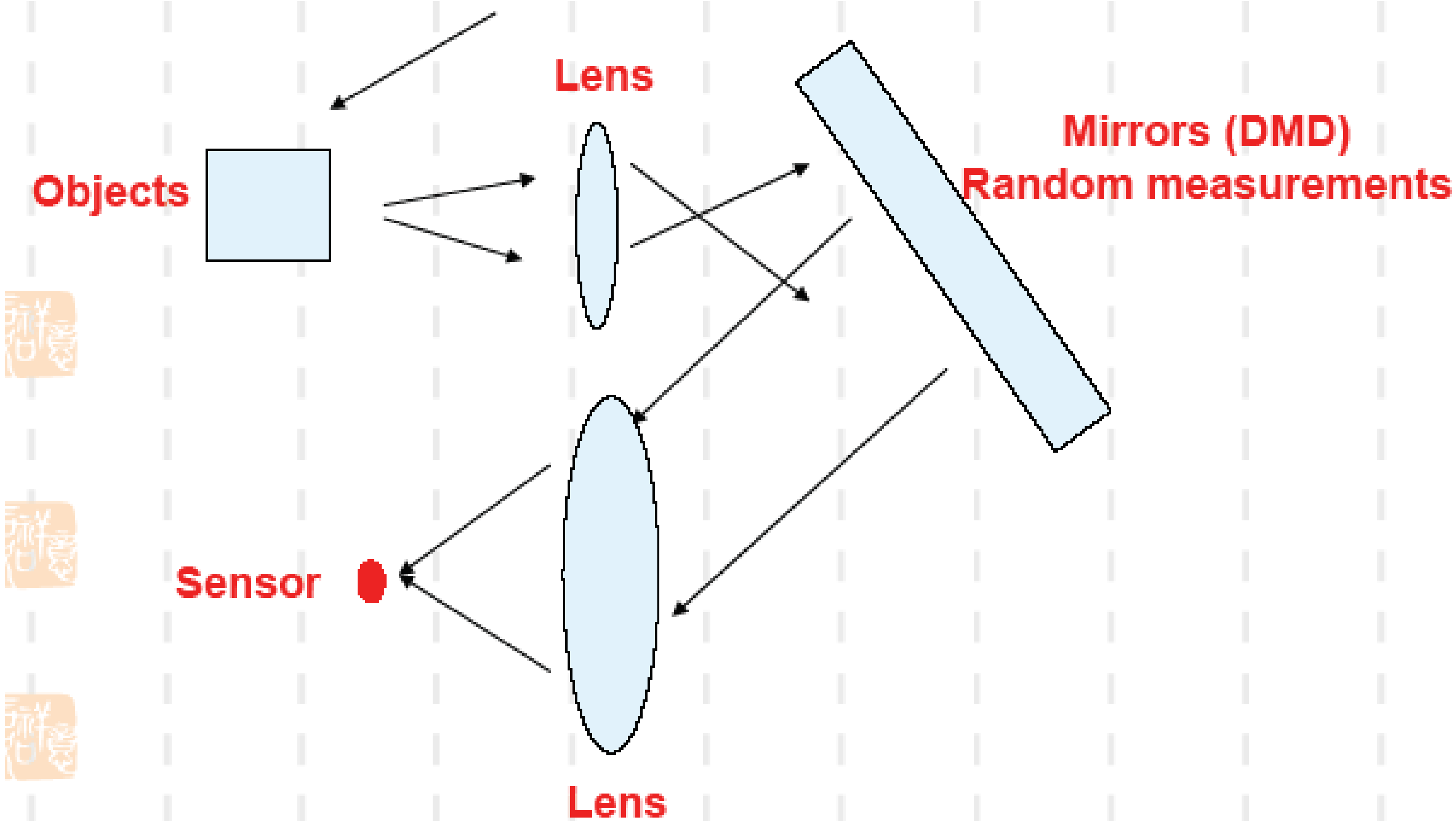
$$x_{p+1} = S_\tau(x_p + \Phi^T(y - \Phi x_p), \Psi).$$

$$S_\tau(f, \Psi) = \sum_\mu \tau(c_\mu(f))\, \psi,$$

$$\tau_h(x) = \begin{cases} x, & |x| \geq \sigma, \\ 0, & |x| < \sigma, \end{cases}$$

Some Applications

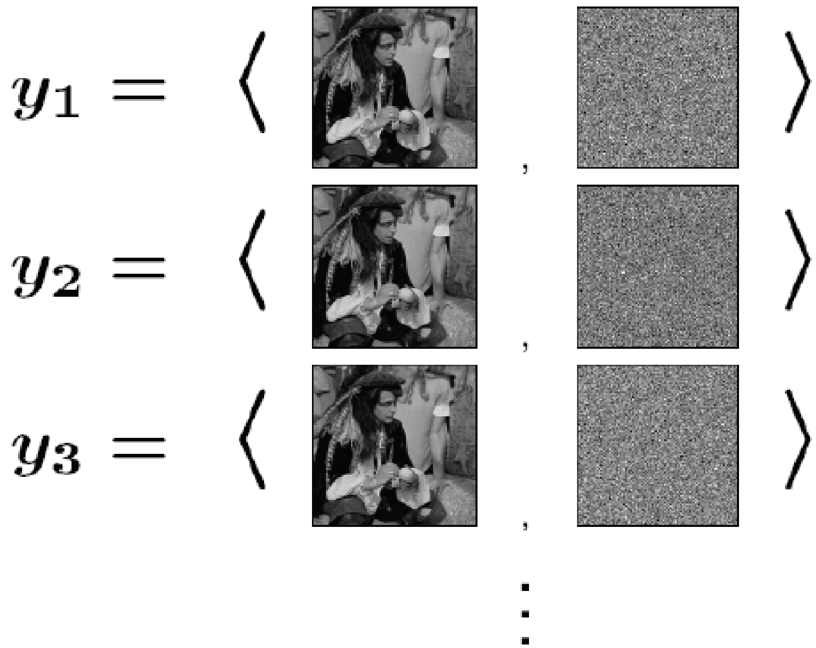
Single-pixel imaging



Some Applications

Single-pixel imaging

Signal is local, measurements are global. Each measurement picks up a little information about each component.

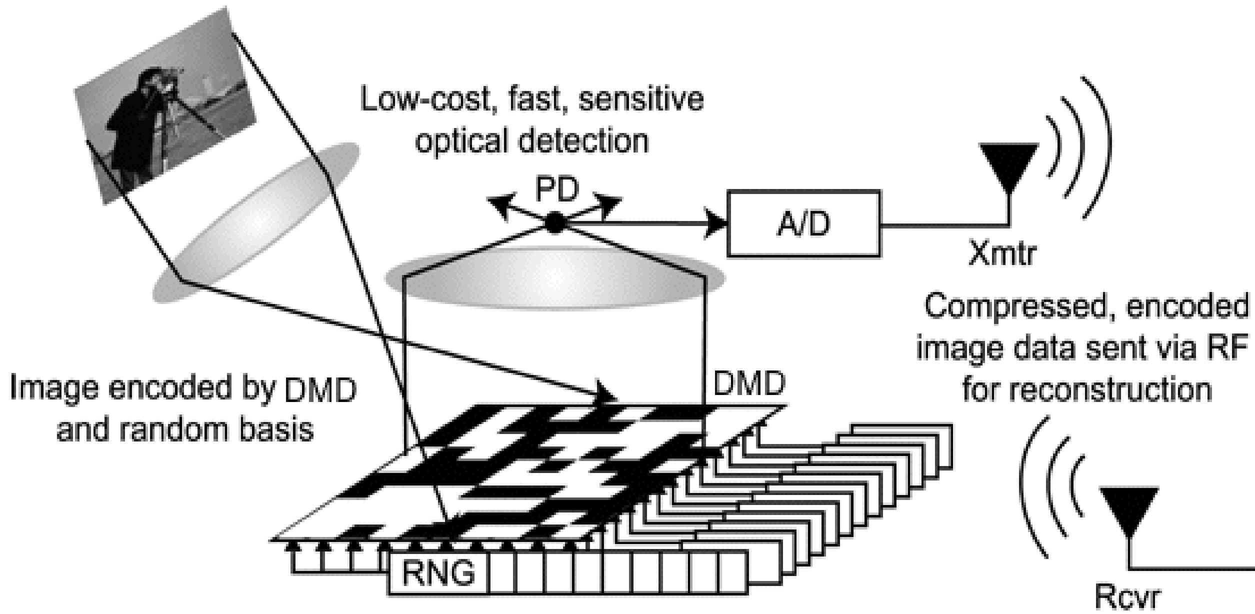


- Single-pixel/sensor but multi-times (SPMT) imaging

Some Applications

Compressive Imaging: A New Single-Pixel Camera

Richard Baraniuk et al., Rice University, 2006



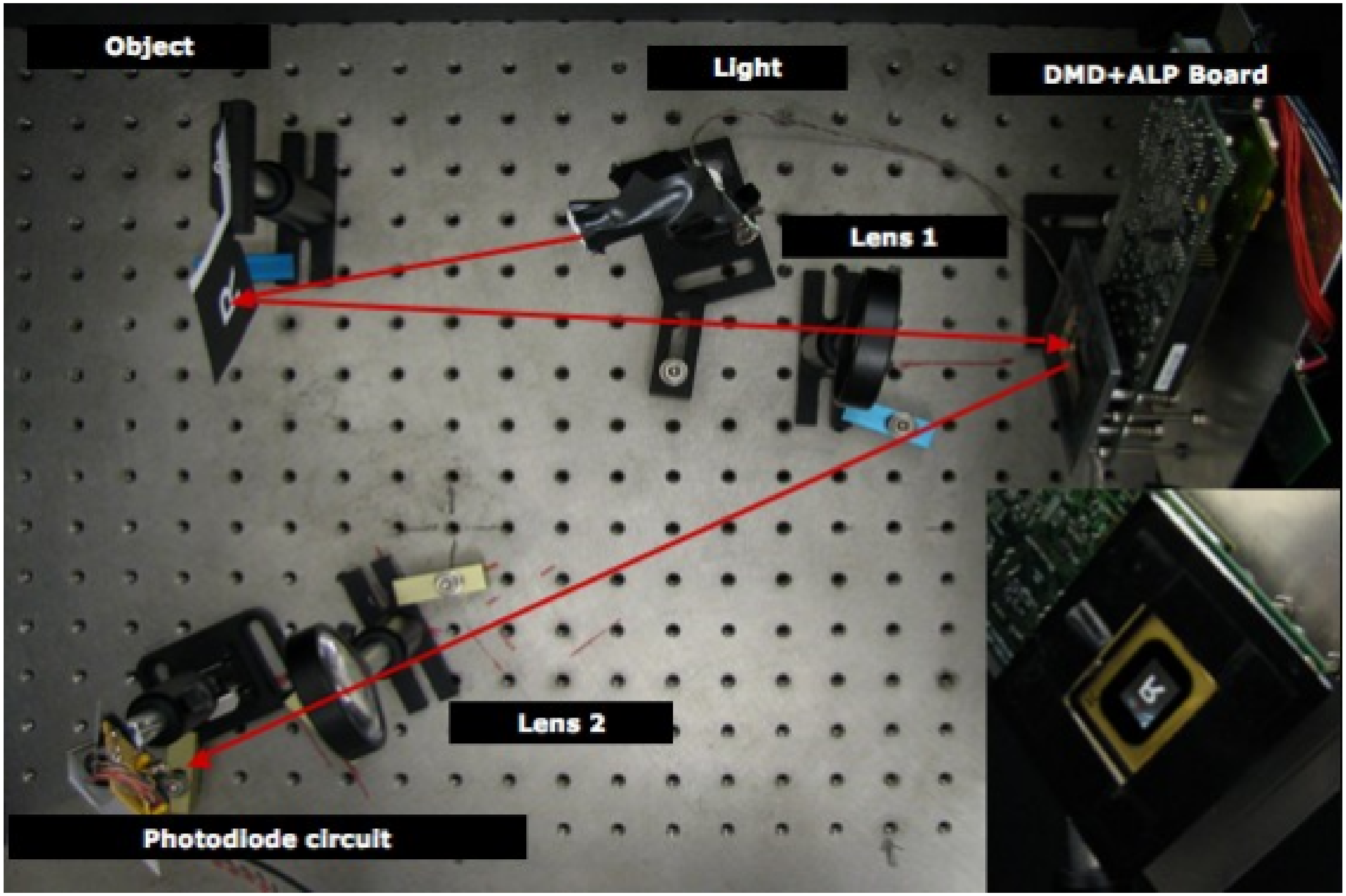
http://www.dsp.ece.rice.edu/cscamera

http://www.dsp.ece.rice.edu/cscamera/inthenews

http://www.tempointeraktif.com/hg/it/2006/10/05/brk,20061005-85452,id.html

Some Applications

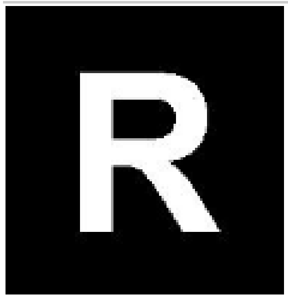
Compressive Imaging: A New Single-Pixel Camera



Some Applications

Compressive Imaging: A New Single-Pixel Camera

RESULTS



Original

16384 Pixels
1600 Measurements
(10%)

16384 Pixels
3300 Measurements
(20%)

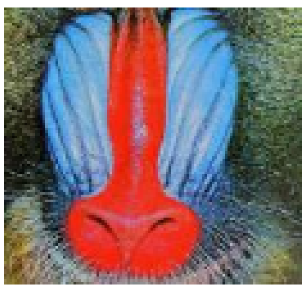
65536 Pixels
1300 Measurements
(2%)

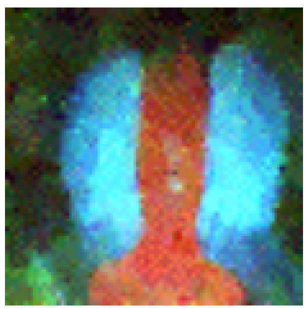
65536 Pixels
3300 Measurements
(5%)

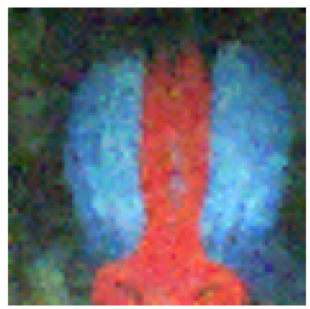
Some Applications

Compressive Imaging: A New Single-Pixel Camera

RESULTS





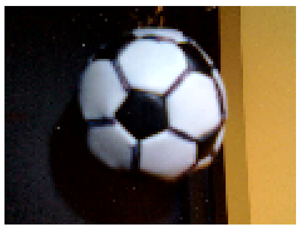
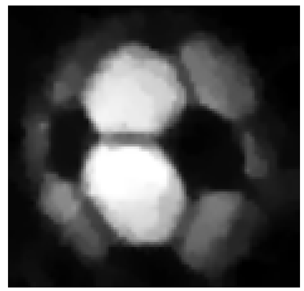
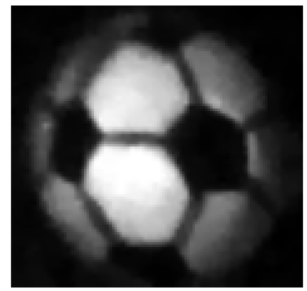


Original	4096 Pixels 800 Measurements (20%)	4096 Pixels 1600 Measurements (40%)	65536 Pixels 6600 Measurements (10%)

Some Applications

Compressive Imaging: A New Single-Pixel Camera

RESULTS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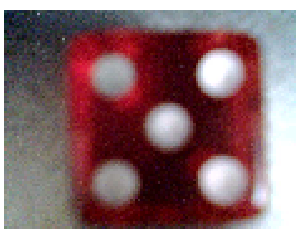	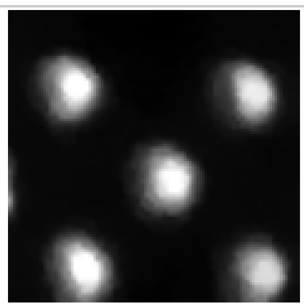	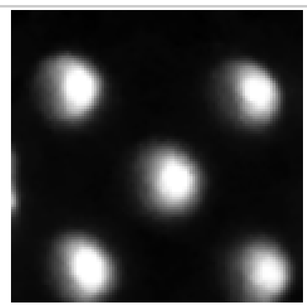
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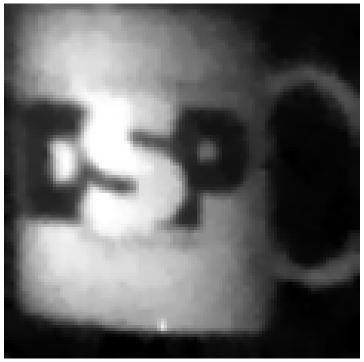
Some Applications

Compressive Imaging: A New Single-Pixel Camera

RESULTS



Original Object	4096 Pixels 800 Measurements (20%)	4096 Pixels 1600 Measurements (40%)

Some Applications

Recover from incomplete Fourier measurement

Another area where compressive sensing has promise is magnetic resonance imaging—where the whole story began. The imperative in MRI is not so much compressing data for storage but acquiring it quickly, because the patient must hold still while an image is formed. Ordinary after-the-fact compression is no help in this respect, but compressive sensing offers hope of faster scanning without loss of resolution or contrast.

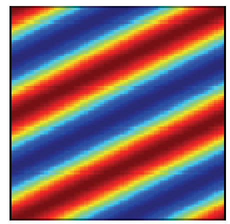
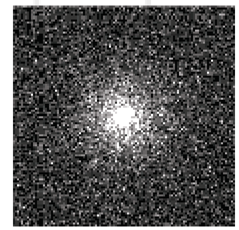
- For random Fourier measurement (e.g., MRI in medical imaging), we only need

$$O(S \cdot \log(N/S))$$

for S-sparse $N \times N$ images.

Some Applications

Recover from incomplete Fourier measurement

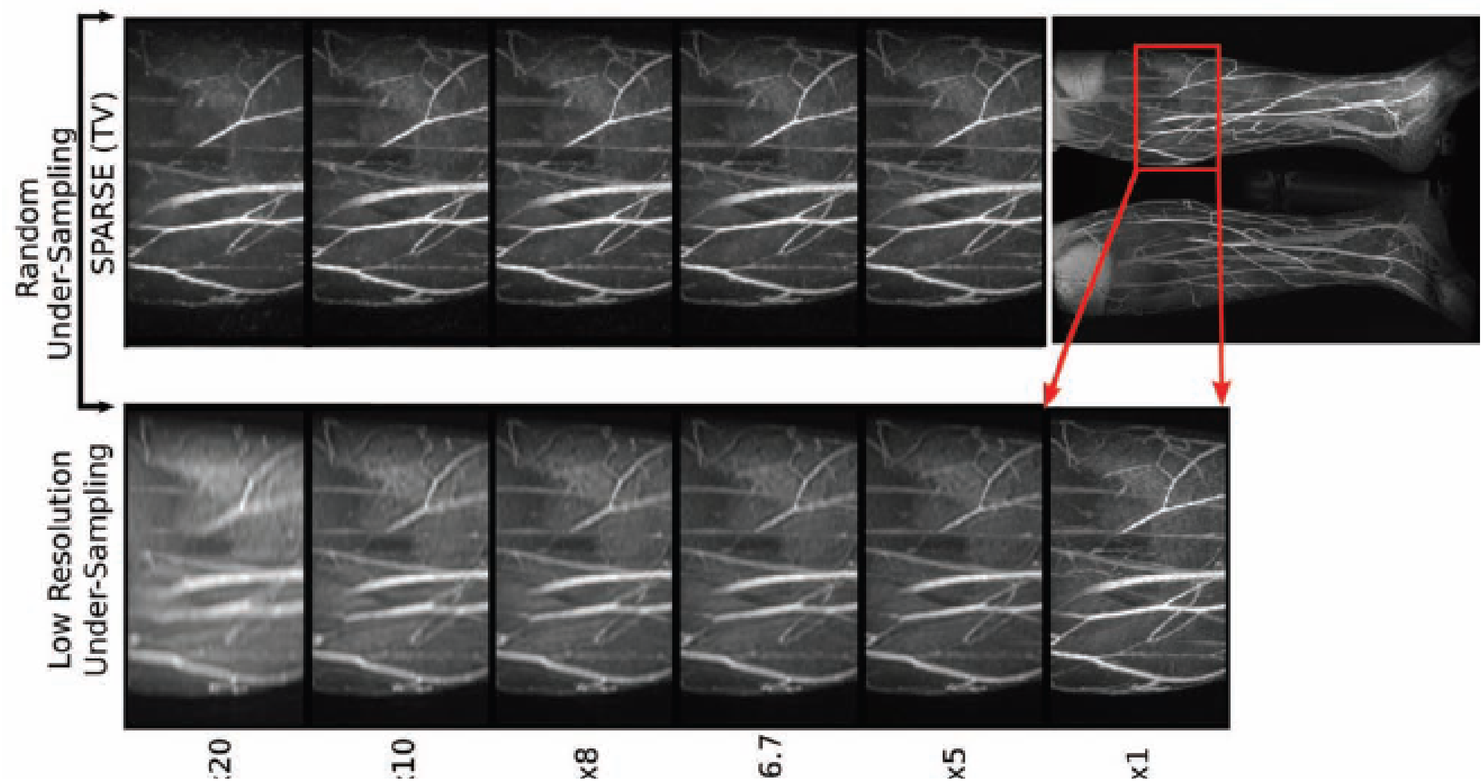
$y_m = \langle$  ,  $\rangle \cdot$ 

- Multi-pixel but one-times (MPOT) imaging.

The measurements take a trade-off between the space and time.

Some Applications

Angiography



An angiogram. From bottom to top, the angiogram is progressively undersampled by larger and larger factors. With a Shannon-Nyquist sampling strategy, the image degrades as the degree of undersampling increases. With compressed sensing, the image remains very crisp even at 20-fold undersampling. The approach used here and in Figure 5 is not l_1-minimization but l_1-minimization of the spatial gradient.

Some Applications

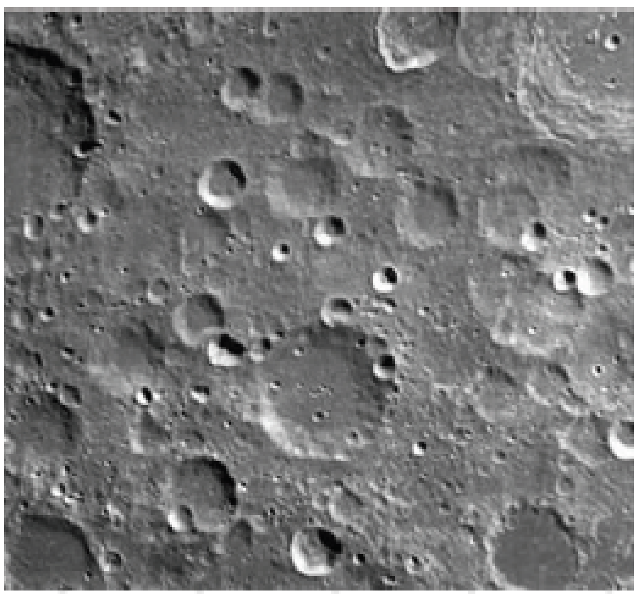
Significance for satellite remote sensing

- On-board encoding: compressed sensing
- Off-line decoding: iterative thresholding

Shift the imaging cost to off-line computational cost

Some Applications

25% measurements in remote sensing



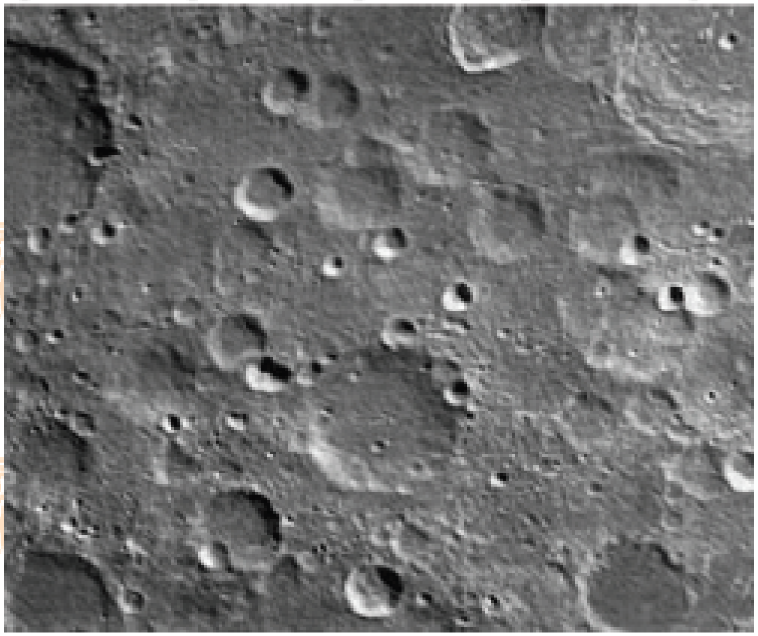
SNR = 38.80 dB

Wavelet-TV

Some Applications

25% measurements in remote sensing

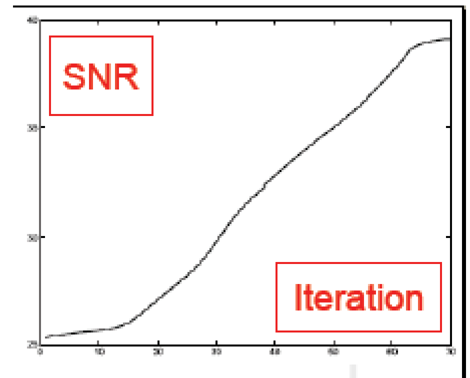
SNR = 39.54 dB



Iterative curvelet thresholding

Some Applications

25% measurements



SNR = 26.07 dB

SNR = 39.11 dB

Wavelet-TV [Lustig et al.] **Iterative curvelet thresholding**

Other Applications

Jianwei Ma and Francois-Xavier Le Dimet, Deblurring from highly incomplete measurements for remote sensing. (IEEE Trans. Geoscience and Remote Sensing, 47 (3), 792-802, 2009).

H. Jung, J. C. Ye, Performance evalution of accelerated functional MRI acquisition using compressed sensing. (Proc. IEEE International Symposium on Biomedical Imaging (ISBI), pp. 702-705, June 28-July 1, 2009, Boston, USA).

J. Provost, F. Lesage, The application of compressed sensing for photo-acoustic tomography. (IEEE Trans Med Imaging, 28(4):585-94, April 2009).

For more papers about CS :

http://www.dsp.ece.rice.edu/cs

Open Problems

- How to build the optimal measurement matrix?
- How to construct the optimal recovery algorithm?

References

[1] E.J. Candès and Michael B.Wakin. **An Introduction To Compressive Sampling** **;** *IEEE SIGNAL PROCESSING MAGAZINE, March 2008*.

[2] Petros Boufounos, Justin Romberg and Richard Baraniuk. **Compressive Sensing : Theory and Applications.**

[3] Yonina Eldar. **Beyond Bandlimited Sampling** ; *EUSIPCO, 2008.*

[4] Justin Romberg and Michael Wakin. **Compressed Sensing: A Tutorial** ; *IEEE Statistical Signal Processing Workshop, Madison, Wisconsin, August 26, 2007.*

[5] **Compressed Sensing Makes Every Pixel Count** ; *What's Happening in The Mathematical Sciences.*

[6] Richard G. Baraniuk. **Compressive Sensing** **;** *Lecture Notes*.

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