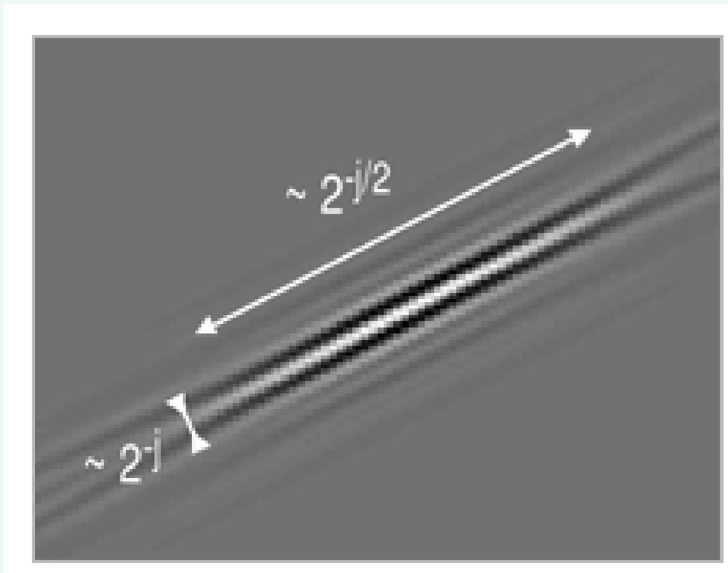
BEYOND WAVELET
DIRECTIONAL MULTIRESOLUTION IMAGE REPRESENTATION : CURVELET TRANSFORM



ENDRA

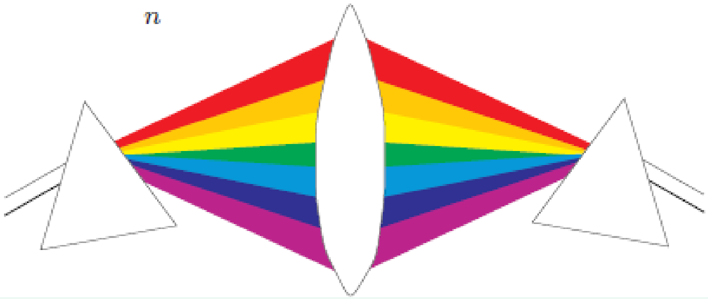
Background

- **Fundamental Question : Parsimonious Representation of Visual Information**

- **Mathematical Foundation: Sparse Representations**

Fourier, Wavelets... $=$ construction of bases for signal expansions:

$$f = \sum_n c_n \psi_n, \quad \text{where} \quad c_n = \langle f, \psi_n \rangle.$$



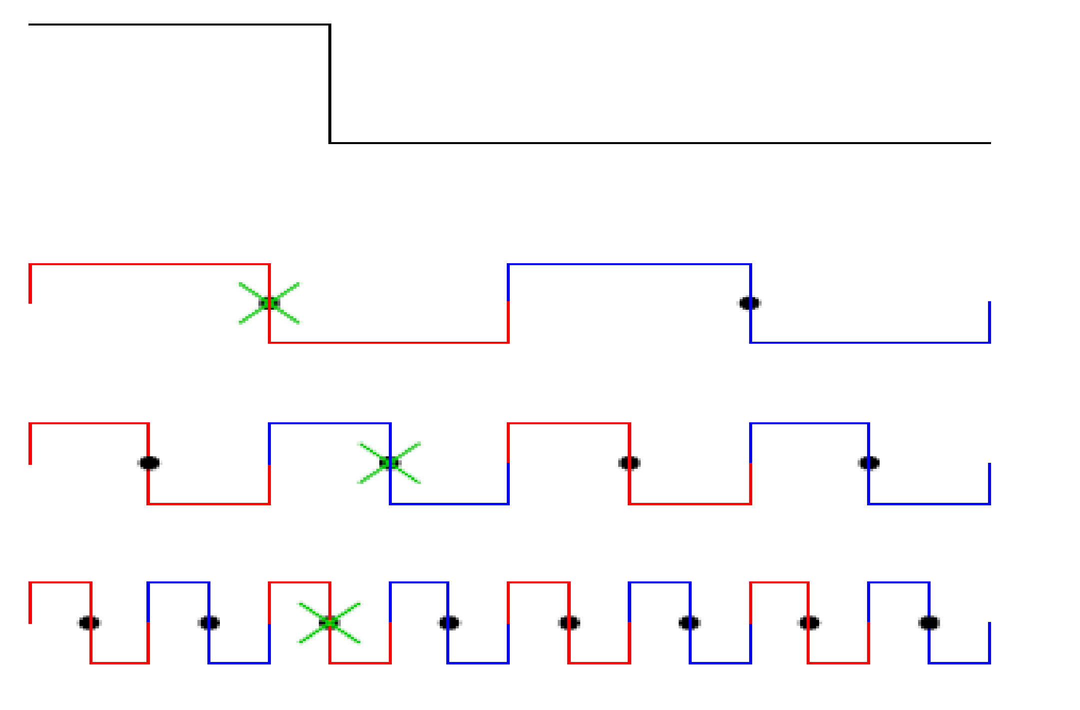
Non-linear approximation:

$$\hat{f}_M = \sum_{n \in I_M} c_n \psi_n, \qquad \text{where } I_M : \text{indexes of biggest } M \text{ coefficients.}$$

Sparse representation: How fast $\|f - \hat{f}_M\| \to 0$ as $M \to \infty$
(e.g. $\|f - \hat{f}_M\|_2^2 \leq C M^{-\alpha}$).

The Success of Wavelets

- Wavelets provide a sparse representation for piecewise smooth signals.

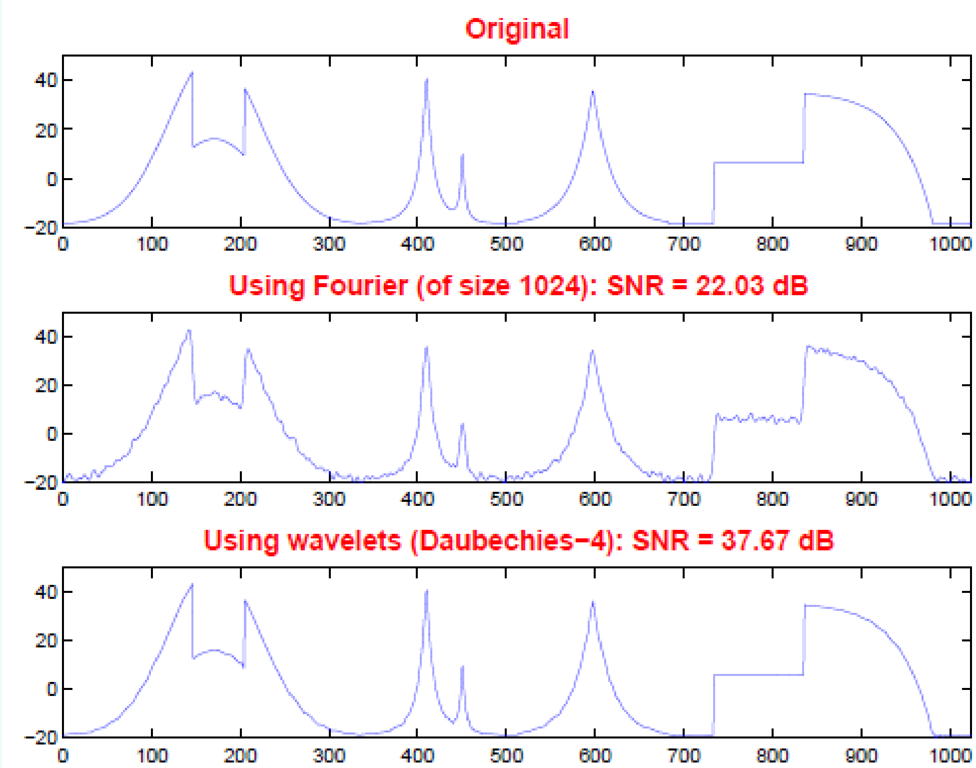


- Multiresolution, tree structures, fast transforms and algorithms, etc.
- Unifying theory $\Rightarrow$ fruitful interaction between different fields.

The Success of Wavelets

Fourier vs. Wavelets

Non-linear approximation: $N = 1024$ data samples; keep $M = 128$ coefficients



The Success of Wavelets

Is This the End of the Story?

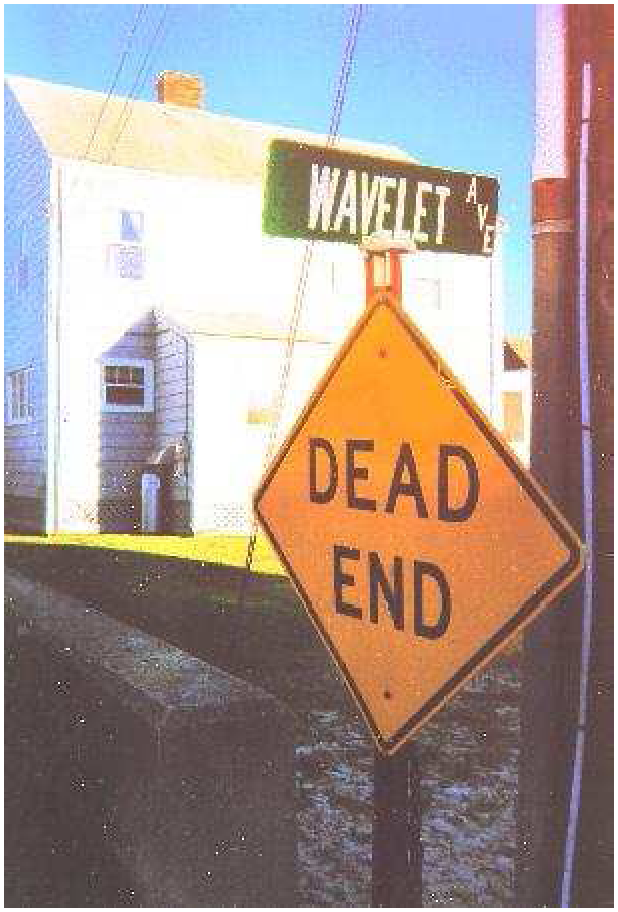


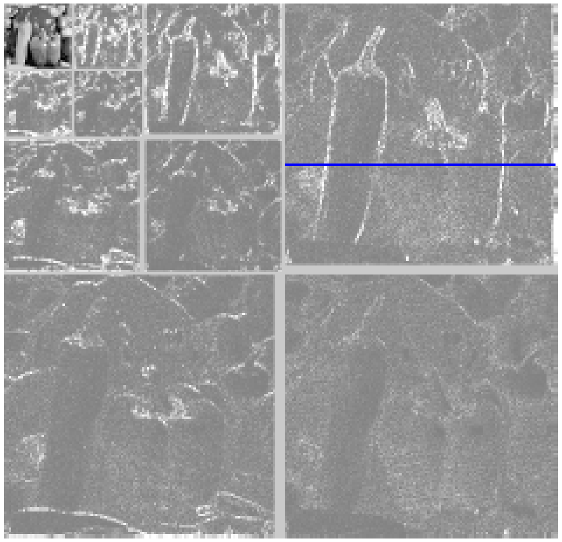
Photo: Ilya Pollak

The Failure of Wavelets

1D discontinuity (Wavelet fails)

• In 1-D: Wavelets are well adapted to abrupt changes or singularities.

• In 2-D: Separable wavelets are well adapted to point-singularities (only).
But, there are (mostly) line- and curved-singularities...



"Wish List" for New Image Representations

- Multiresolution ... successive refinement
- Localization ... both space and frequency
- Critical sampling ... correct joint sampling
- Directionality ... more directions
- Anisotropy ... more shapes

Our emphasis is on discrete framework that leads to algorithmic implementations.

Edges – discontinuities across curves

- **Synthesis: edge location is known in advance, representation adapted respectively;**
- **Analysis: edge location is unknown; two approaches arise:**
    - **Adaptive (Lagrangian representation – constructed using the full knowledge of the structure and adapting to the structure perfectly);**
    - **Non-adaptive (Eulerian – fixed, constructed once and for all).**

Surprise:

Despite common belief that adaptive representation is essentially more powerful than fixed non-adaptive, it turns out that there is a fixed non-adaptive technique essentially as good as adaptive representation from the point of view of asymptotic m-term approximation errors. [1]

Why do we need curvelet?

- *The comparing of the approximation using the best m nonzero terms*

Fourier method $\left\| f - \tilde{f}_m^{F} \right\|_2^2 \approx m^{-1/2}, m \to \infty$

Wavelet method $\left\| f - \tilde{f}_m^{W} \right\|_2^2 \approx m^{-1}, m \to \infty$

Adaptive method $\left\| f - \tilde{f}_m^{A} \right\|_2^2 \approx m^{-2}, m \to \infty$

Curvelet method $\left\| f - \tilde{f}_m^{C} \right\|_2^2 \le C \bullet m^{-2} (\log m)^3, m \to \infty$

- **Wavelet failed in the presence of curve discontinuity.**
- **Adaptive methods are ideal but difficult to implement.**
- **Non-adaptive methods (curvelet) can represent the ideal behavior of an adaptive representation.**

Development History of Curvelet

- 1998, Ridgelets, Dr. Candès. [2], [3]
- 1999, Curvelet 99, Dr. Candès and Dr. Donoho. [1], [4]
- 2002, second generation curvelets. [5]
- 2002-present Curvelet.org, Fast discrete curvelet transform, Curvelab, 3D Discrete Curvelet Transform.[6],[7],[8].

Comparison to Fourier & Wavelet

- 1807, J.B. Fourier:
  - All periodic functions can be expressed as a weighted sum of trigonometric function
  - Denied publication by Lagrange, Legendre and Laplace
  - 1822: Fourier's work is finally published
  - …
  - …
  - …
  - …
  - 1965, Cooley & Tukey: Fast Fourier Transform

143 years

Comparison to Fourier & Wavelet

- 1911: Haar
- 1930: Littlewood Paley
- 1940: Gabor
- 1960: Calderón-Zygmund

- 1980's beginnings of wavelets in physics, vision, speech processing (ad hoc)
- … little theory … why/when do wavelets work?
- 1986 Mallat unified the above work
- 1985 Morlet & Grossman continuous wavelet transform … asking: how can you get perfect reconstruction without redundancy?

Comparison to Fourier & Wavelet

- 1985 Meyer tried to prove that no orthogonal wavelet other than Haar exists, found one by trial and error!
- 1987 Mallat developed multiresolution theory, DWT, wavelet construction techniques (but still noncompact)
- 1988 Daubechies added theory: found compact, orthogonal wavelets with arbitrary number of vanishing moments!

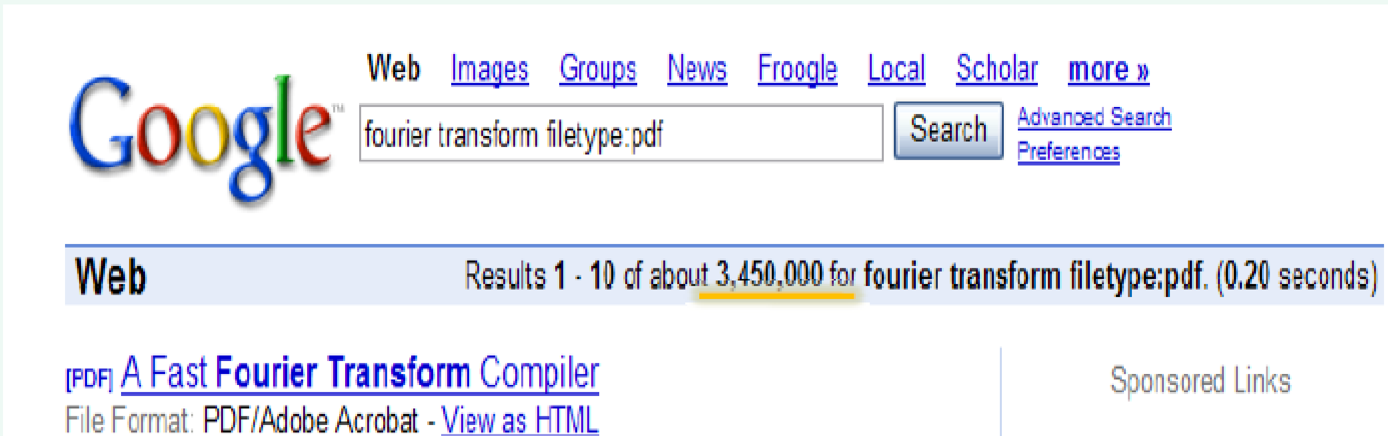
- **1990: biorthogonal wavelets**

- 1990's: wavelets took off, attracting both theoreticians and engineers

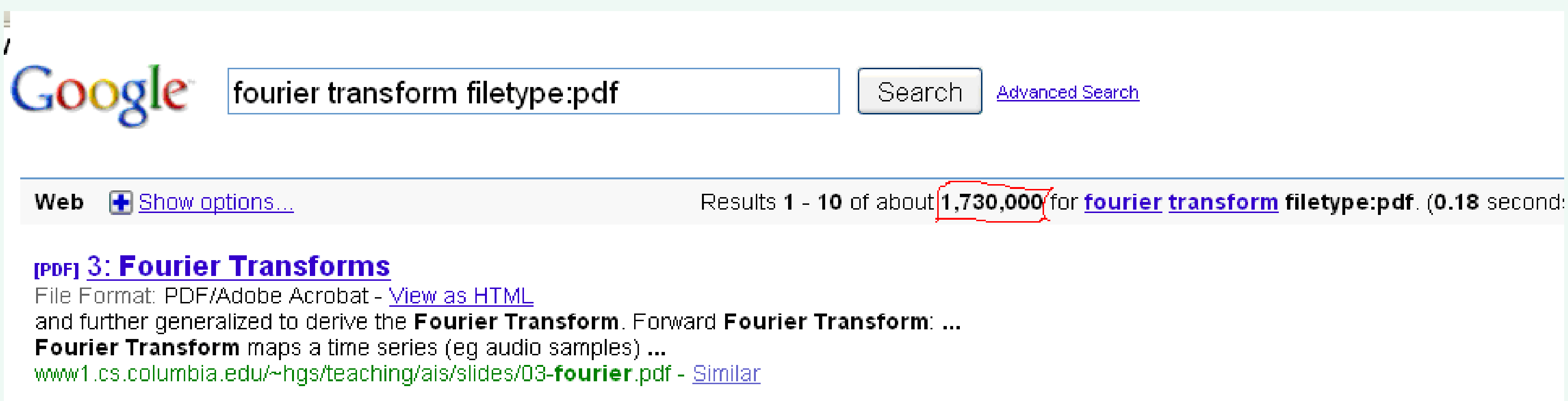
- **1994: second generation wavelets**

Comparison to Fourier & Wavelet

2006

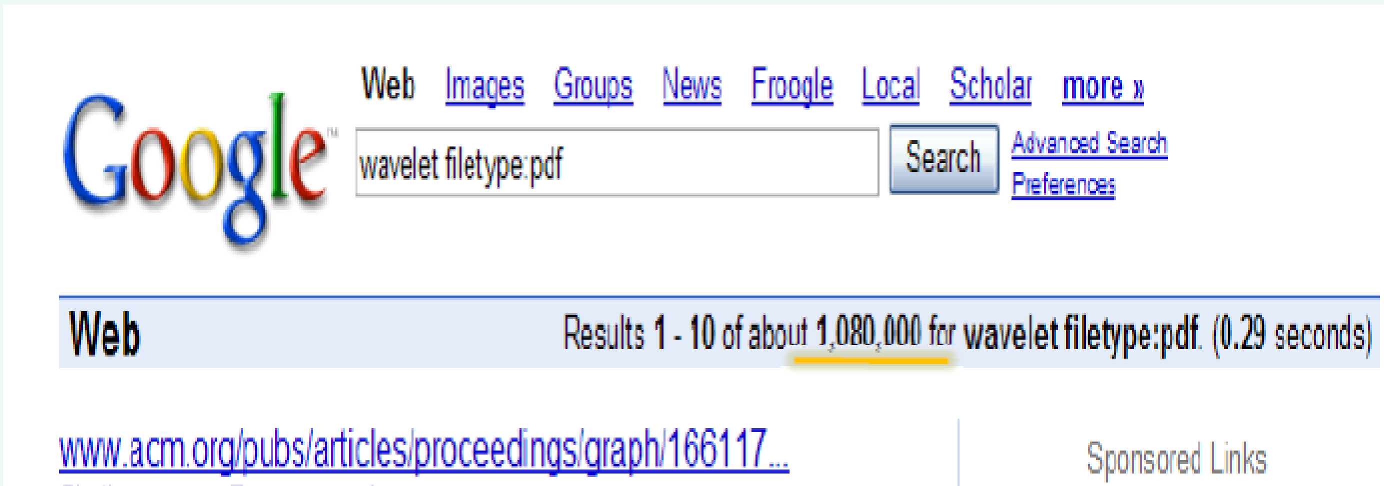


2009

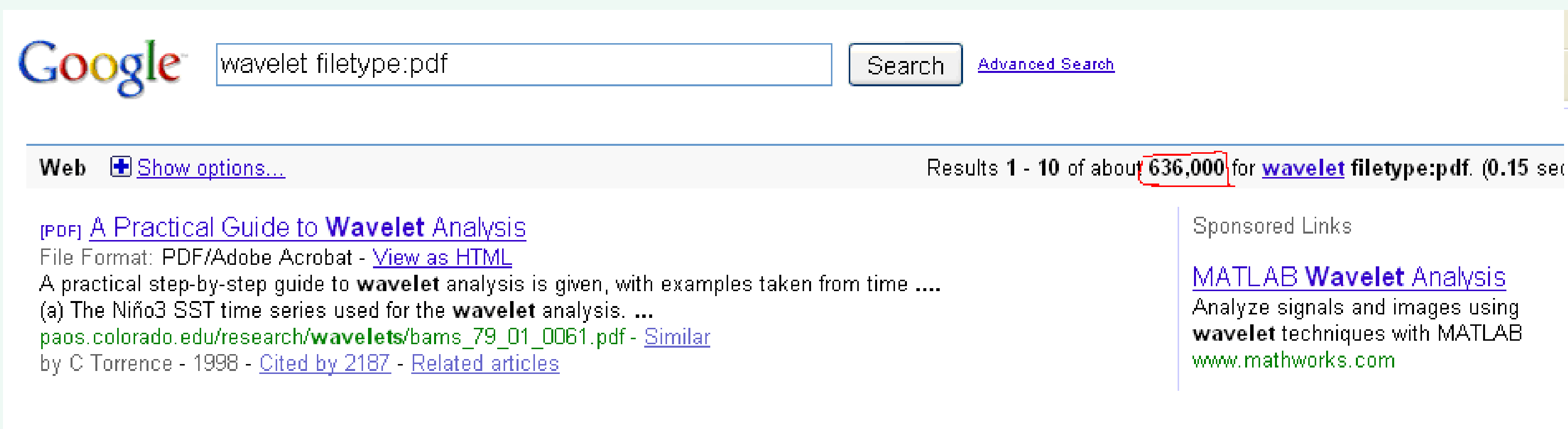


Comparison to Fourier & Wavelet

2006

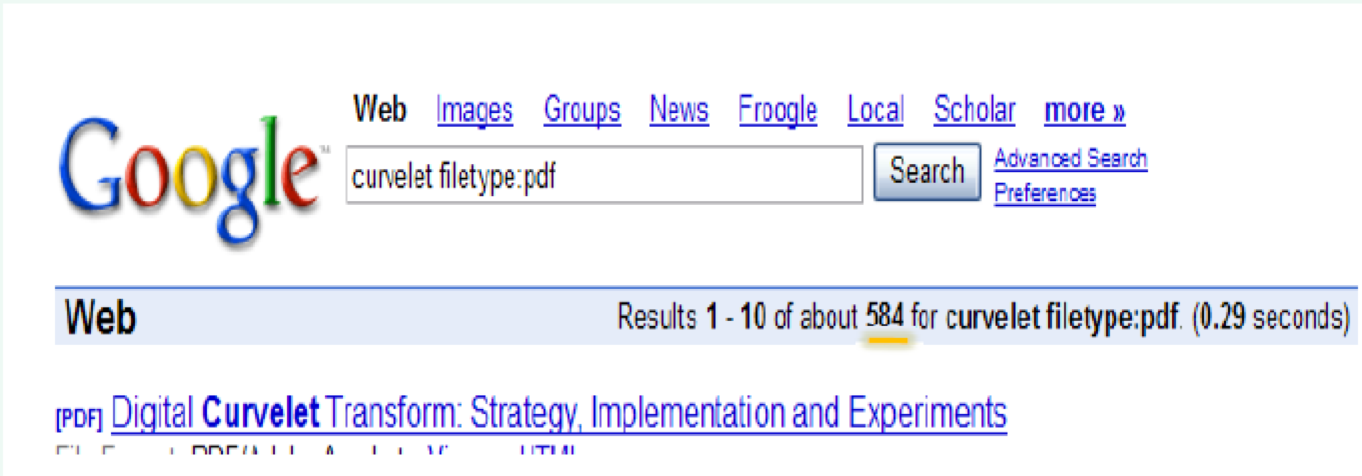


2009

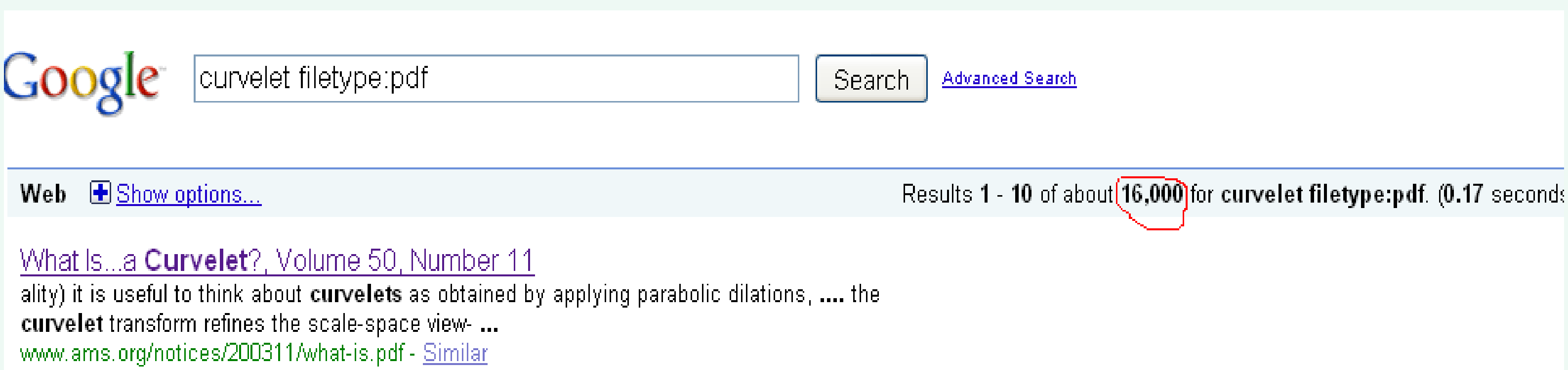


Comparison to Fourier & Wavelet

2006



2009



Comparison to Fourier & Wavelet

2009

Books › **"fourier"**

Related Searches: charles fourier.

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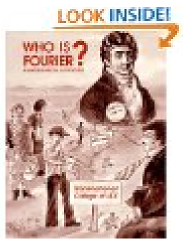
1. 

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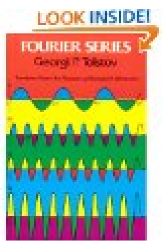
2. 

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A Student's Guide to Fourier Transforms: With Applications in Physics and Engineering by J. F. James (**Paperback** - Feb 10, 2003)
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Comparison to Fourier & Wavelet

2009

Books › "wavelet"

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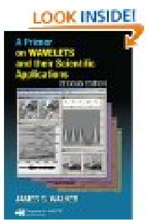
1. 

A Primer on Wavelets and Their Scientific Applications, Second Edition (Studies in Advanced Mathematics) by James S. Walker (**Paperback** - Jan 29, 2008)

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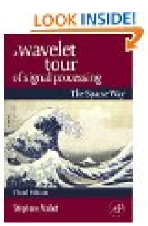
2. 

A Wavelet Tour of Signal Processing, 3rd ed., Third Edition: The Sparse Way by Stephane Mallat (**Hardcover** - Dec 25, 2008)

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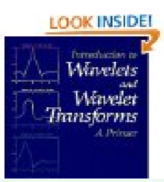
Excerpt - page 1: "... to all signals is a hopeless quest. The discovery of **wavelet** orthogonal bases and local time-frequency dictionaries has opened the door ..."
Surprise me! See a random page in this book.

3. 

Introduction to Wavelets and Wavelet Transforms: A Primer by C. Sidney Burrus, Ramesh A. Gopinath, and Haitao Guo (**Paperback** - Aug 24, 1997)

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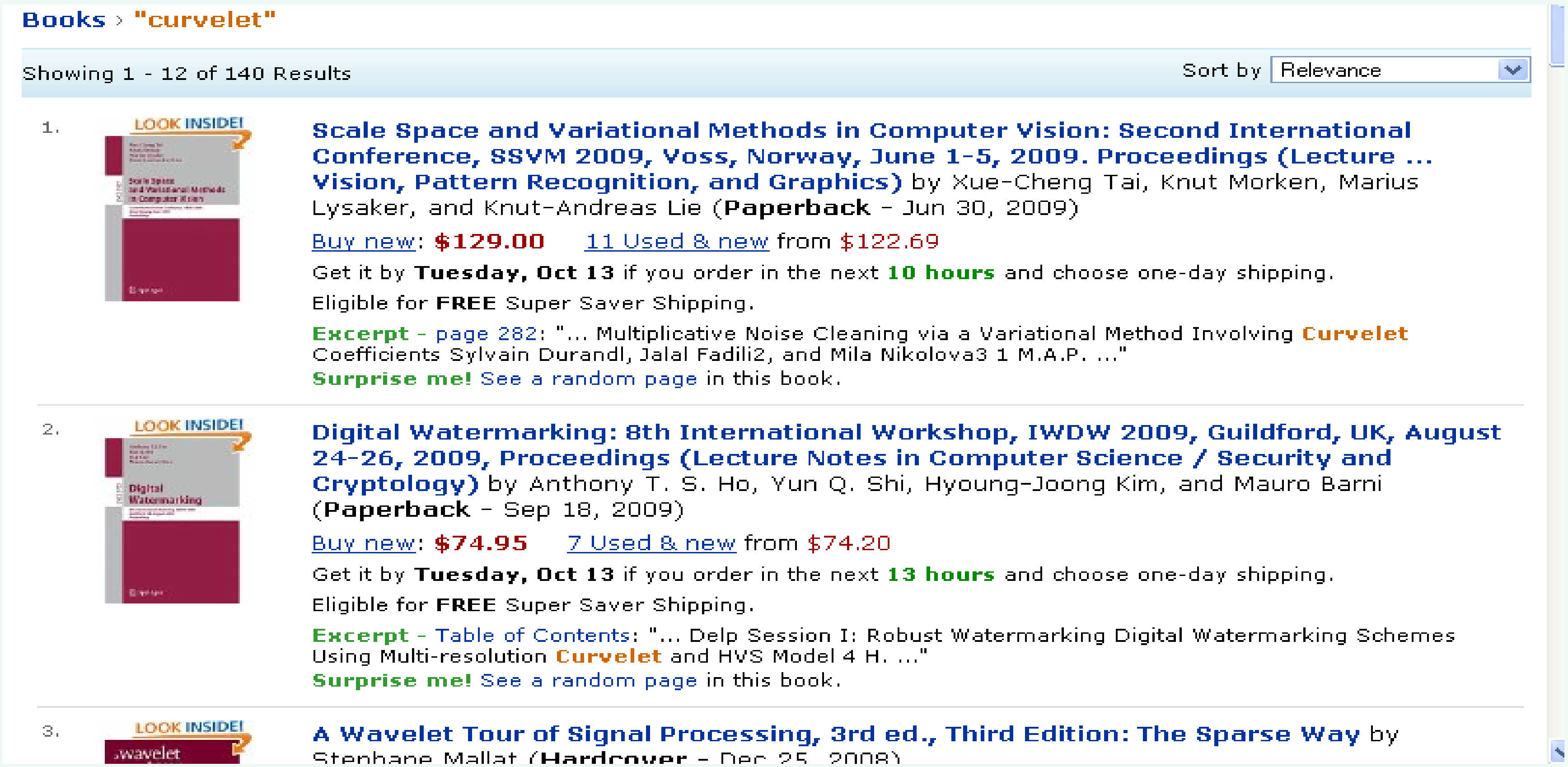
Comparison to Fourier & Wavelet

2009



Books › **"curvelet"**

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1. **Scale Space and Variational Methods in Computer Vision: Second International Conference, SSVM 2009, Voss, Norway, June 1-5, 2009. Proceedings (Lecture ... Vision, Pattern Recognition, and Graphics)** by Xue-Cheng Tai, Knut Morken, Marius Lysaker, and Knut-Andreas Lie (**Paperback** - Jun 30, 2009)
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   Excerpt - page 282: "... Multiplicative Noise Cleaning via a Variational Method Involving **Curvelet** Coefficients Sylvain Durandl, Jalal Fadili2, and Mila Nikolova3 1 M.A.P. ..."
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2. **Digital Watermarking: 8th International Workshop, IWDW 2009, Guildford, UK, August 24-26, 2009, Proceedings (Lecture Notes in Computer Science / Security and Cryptology)** by Anthony T. S. Ho, Yun Q. Shi, Hyoung-Joong Kim, and Mauro Barni (**Paperback** - Sep 18, 2009)
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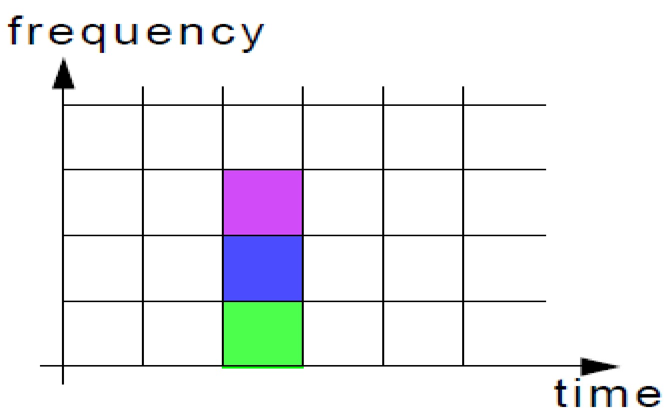
   Excerpt - Table of Contents: "... Delp Session I: Robust Watermarking Digital Watermarking Schemes Using Multi-resolution **Curvelet** and HVS Model 4 H. ..."
   Surprise me! See a random page in this book.

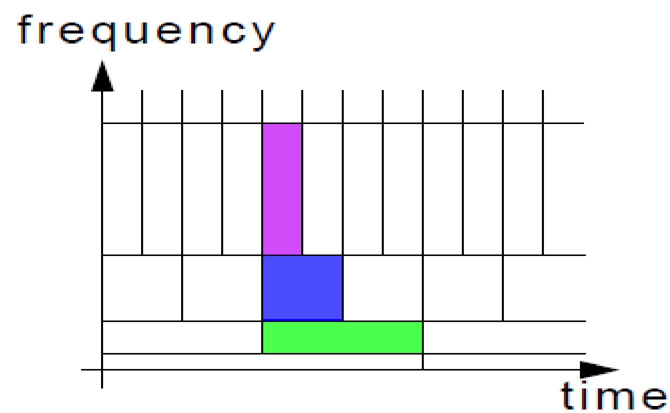
3. **A Wavelet Tour of Signal Processing, 3rd ed., Third Edition: The Sparse Way** by Stephane Mallat (**Hardcover** - Dec 25, 2008)

Comparison to Fourier & Wavelet

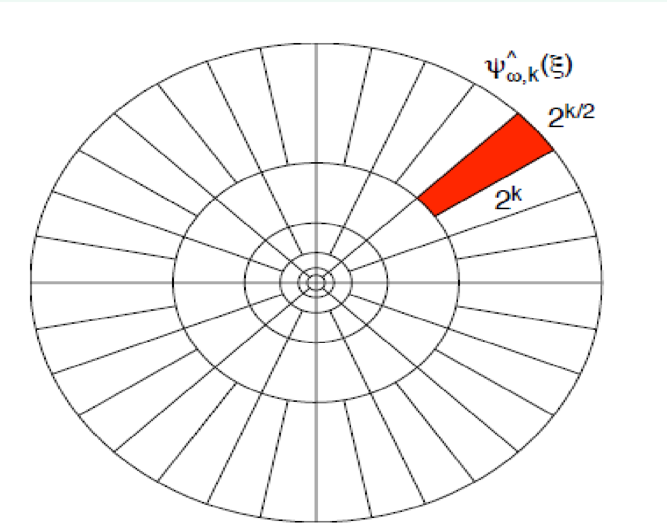
STFT/Gabor



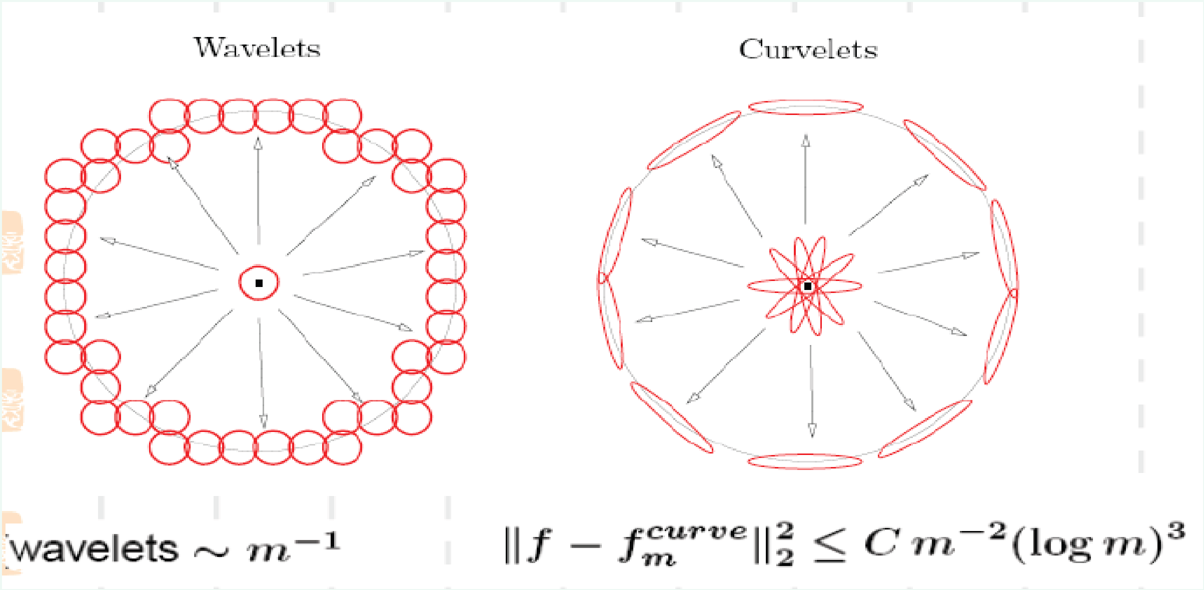
Wavelet

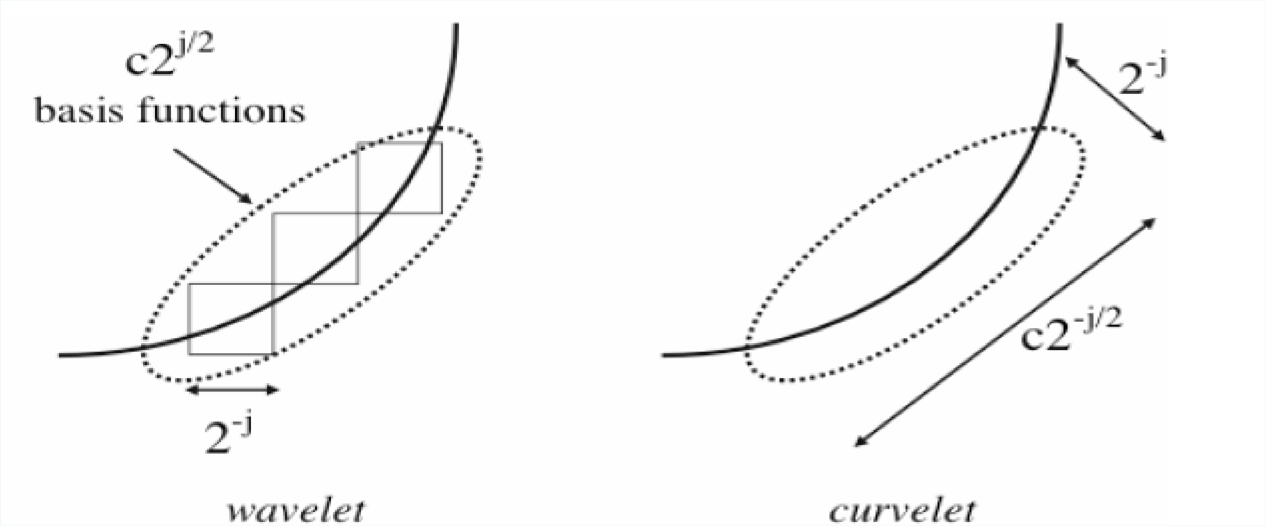


Curvelet



Comparison of Curvelet & Wavelet





Curvelet Transform Technical

The Curvelet Transform includes four stages:

- Sub-band decomposition
- Smooth partitioning
- Renormalization
- Ridgelet analysis

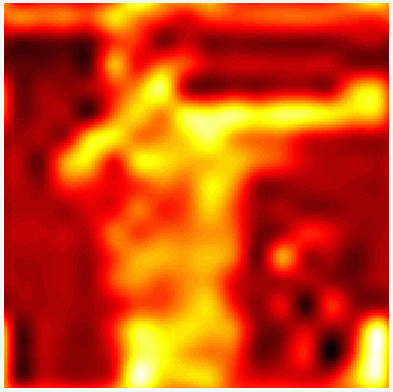
Sub-band Decomposition

$$f \mapsto \left(P_0 f, \Delta_1 f, \Delta_2 f, \ldots\right)$$

f

***P_0* – Low-pass filter.**

Δ_1, Δ_2, … – Band-pass (high-pass) filters.

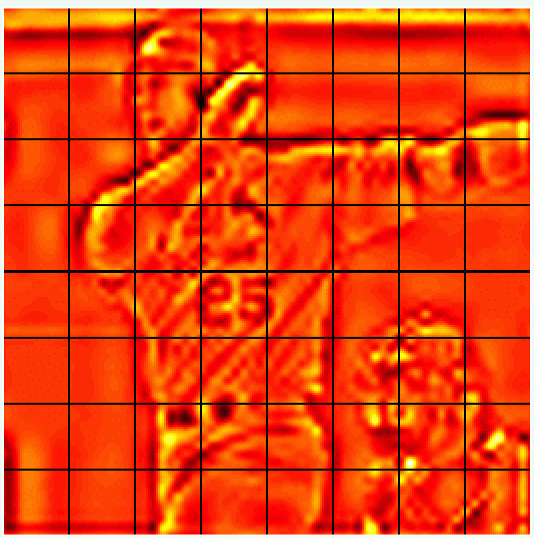


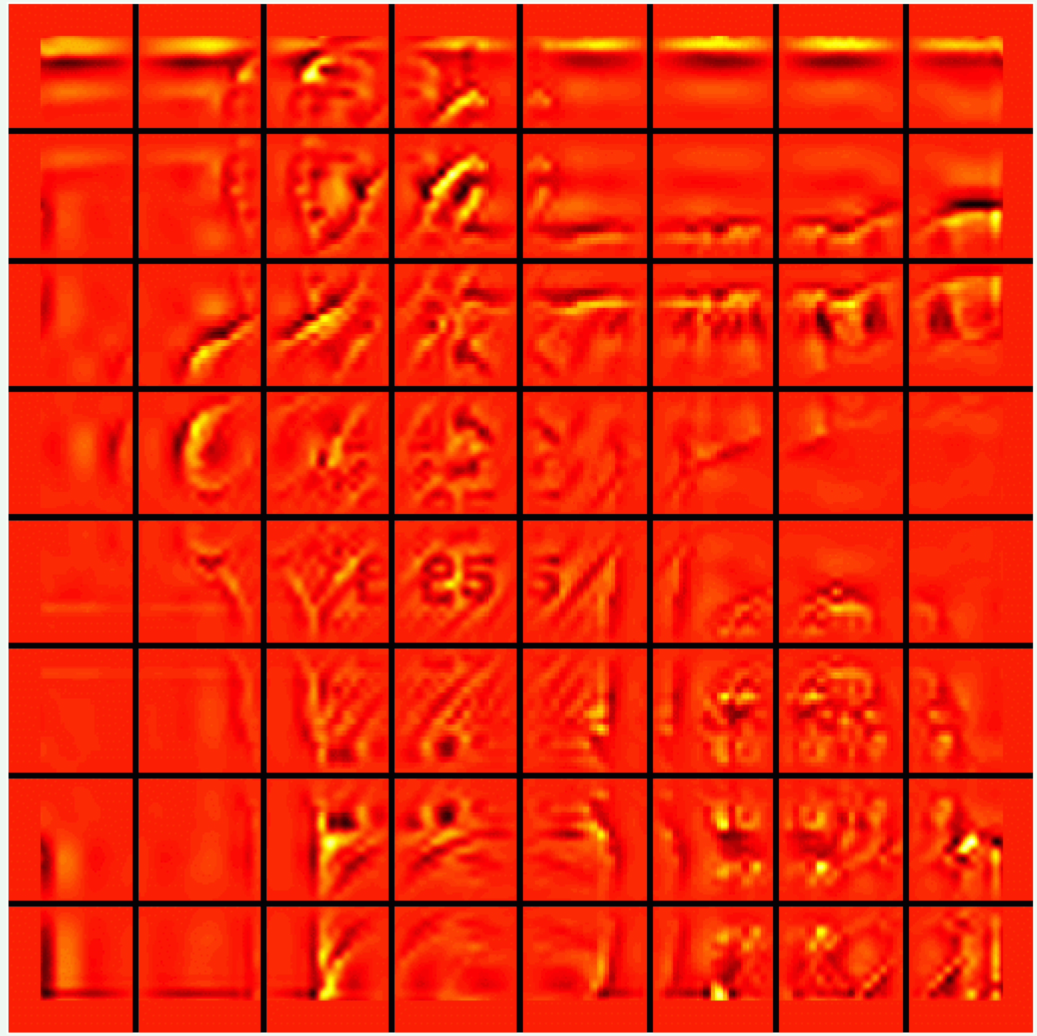
$P_0 f$

$\Delta_1 f$

$\Delta_2 f$

Smooth Partitioning





Smooth Partitioning

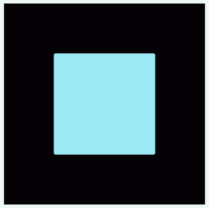
- **Let w be a smooth windowing function with 'main' support of size $2^{-s}\times 2^{-s}$. For each square, w_Q is a displacement of w localized near Q.**

- **Multiplying $\Delta_s f$ with w_Q ($\forall Q \in \mathcal{Q}_s$) produces a smooth dissection of the function into 'squares'.**

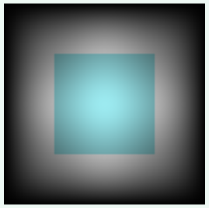
$$h_Q = w_Q \cdot \Delta_s f$$

Example:

An indicator of the dyadic square (but not smooth!!).



Smooth window function with an extended compact support:



Renormalization

- Renormalization is centering each dyadic square to the unit square $[0,1]\times[0,1]$.
- For each Q, the operator T_Q is defined as:

$$\left(T_Q f\right)\left(x_1, x_2\right) = 2^s f\left(2^s x_1 - k_1, 2^s x_2 - k_2\right)$$

- Each square is renormalized:

$$g_Q = T_Q^{-1} h_Q$$

Ridgelet Analysis

- Each normalized square is analyzed in the ridgelet system:

$$\alpha_{(Q,\lambda)} = \left\langle g_Q , \rho_\lambda \right\rangle$$

  - The ridge fragment has an aspect ratio of $2^{-2s} \times 2^{-s}$.
  - After the renormalization, it has localized frequency in band $|\xi| \in [2^s, 2^{s+1}]$.
  - A ridge fragment needs only a very few ridgelet coefficients to represent it.

INVERSE CURVELET TRANSFORM

- Ridgelet Synthesis:

$$g_Q = \sum_{\lambda} \alpha_{(Q,\lambda)} \cdot \rho_{\lambda}$$

- Renormalization:

$$h_Q = T_Q g_Q$$

- Smooth Integration:

$$\Delta_s f = \sum_{Q \in \mathbf{Q}_s} w_Q \cdot h_Q$$

- Sub-band Recomposition:

$$f = P_0 \left(P_0 f \right) + \sum_{s} \Delta_s \left(\Delta_s f \right)$$

FAST DISCRETE CURVELET TRANSFORM (FDCvT)

Suggests two algorithmic strategies, **Unequi-Spaced Fast Fourier Transform (USFFT)** based and **Frequency wrapping based** FDCvT.[7]

CurveLab Toolbox

CurveLab is a collection of Matlab and C++ programs for the Fast Discrete Curvelet Transform in two and three dimensions. [9]

Some Paper About Curvelet Application

1. Vehicle Recognition Based on Fourier, Wavelet and Curvelet Transforms - a Comparative Study, 2007. [10]

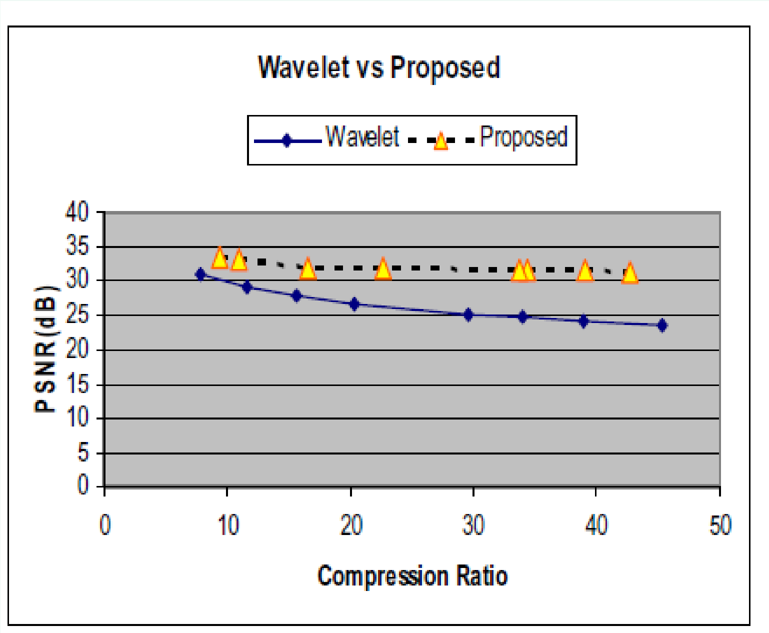
The classifier used in this paper is called k nearest-neighbor.

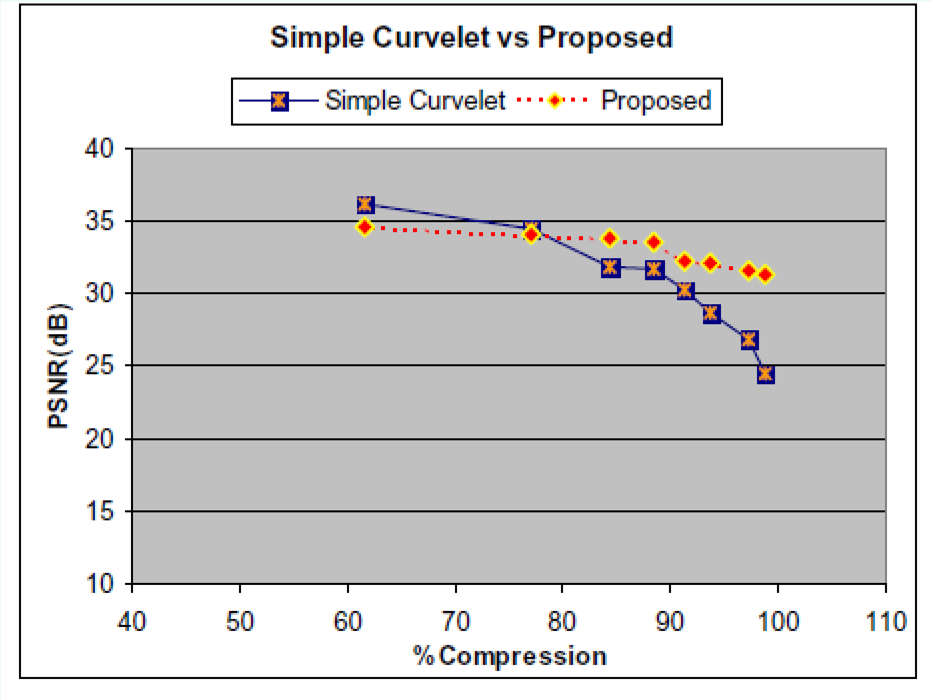
Table 1.The recognition rates with different lengths of 3 various feature vectors.

Number of features (coefficients)	All coefficients (FFT=16384, Wavelet=16384, curvelet=119449)	13130	10000	9000	8000	6000
Recognition rate using FFT	97%	90%	87%	85%	85%	70%
Recognition rate using Wavelet	92%	89%	89%	87%	90%	85%
Recognition rate using Curvelet	100%	100%	97%	97%	95%	95%

During the next works, we will use of the other classifiers to improve our system recognition rate.

2. Curvelet-based Image Compression with SPIHT, 2007. [11]





The future work will be on integration of some other encoding techniques with curvelets to analyze the effect on compression ratio and PSNR of the input image.

3. DENOISING OF COMPUTER TOMOGRAPHY IMAGES USING CURVELET TRANSFORM, 2007 [12]

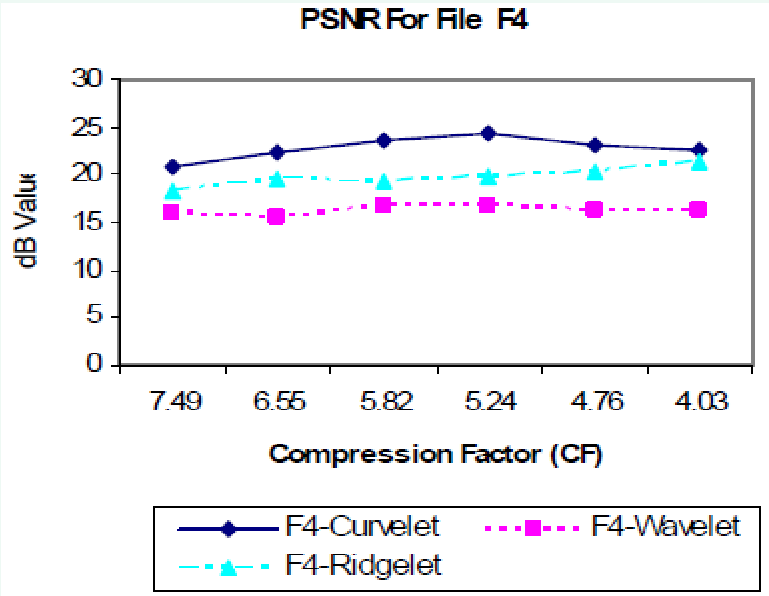
PSNR

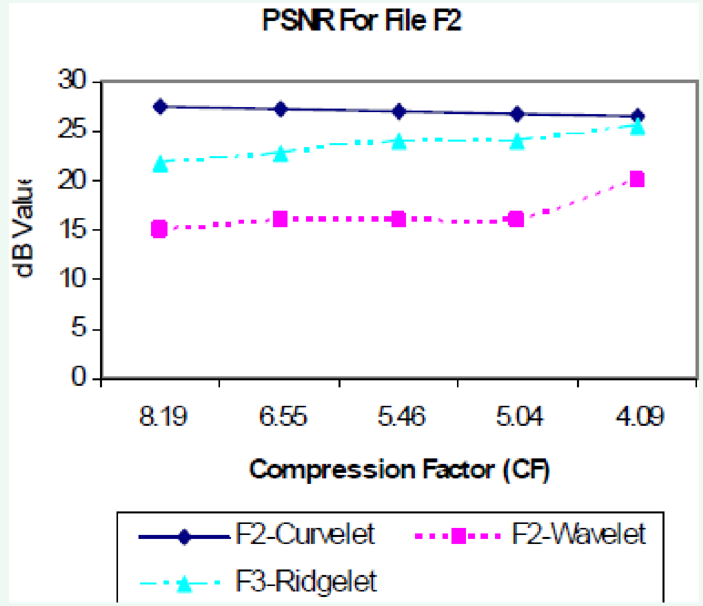
Noise	Mean		Standard Deviation	
	CvT	WT	CvT	WT
Random noise	17.27	14.14	2.81	1.17
Gaussian noise	15.16	13.12	1.03	0.91
Salt & Pepper noise	14.43	16.05	0.62	0.06
Speckle noise	19.65	39.55	21.12	7.06

In all cases it was found that the Curvelet transform outperforms the Wavelet transform in terms of PSNR and the Curvelet denoised images appear visually more pleasant than the Wavelet denoised images.

The Curvelet transform does not effectively remove the Salt and Pepper noise and Speckle noise from the medical images, and so Curvelet transform is not suited for removal of these two noises though it recovers the curves and edges perfectly.

4. Image Compression Using Curvelet, Ridgelet and Wavelet Transform,A Comparative Study, 2008 [13]





Curvelet Transform gives the best performance for PSNR for the files F2, F4 clearly. The quantitative PSNR values in case of Ridgelet Transform are better in case of a few images. But the subjective visual inspection shows that the Curvelet is the best for Compression out of all three transforms.

• **Join in *Curvelet mailing-list* to update information and knowledge about the current researches in curvelet.**

References

[1] E.J. Candès and D.L. Donoho. Curvelets – A Surprisingly Effective Non-adaptive Representation for Objects with Edges; *Curves and Surfaces, L. L. Schumaker et al. (eds), Vanderbilt University Press, Nashville, TN, 1999.*

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[3] E.J. Candès and D.L. Donoho. Ridgelets a key to higher-dimensional intermittency ; *Phil. Trans. R. Soc. Lond. A., 357, 2495—2509, 1999.*

[4] J.L Starck, E.J. Candès and D.L. Donoho. Curvelets, multiresolution representation, and scaling laws ; *IEEE Transactions on Image Processing, 11 670—684, 2000.*

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[5] E.J. Candès and D.L. Donoho. New tight frames of curvelets and optimal representations of objects with piecewise-C^2 singularities; *Comm. Pure Appl. Math., 57 219-266 2002*.

[6] http://curvelet.org/.

[7] E.J. Candès, L. Demanent , D.L. Donoho and L.Ying. Fast Discrete Curvelet Transform; *Curves and Surfaces, L. L. Schumaker et al. (eds), Vanderbilt University Press, Nashville, TN, 2005*.

[8] L.Ying, L. Demanent and E.J. Candès. 3D Discrete Curvelet Transform; Proceedings of SPIE, 591413-591413-11, *2005*.

[9] E.J. Candès, L. Demanent and L.Ying. CurveLab Toolbox, Version 2.0.3.

[10] Farhad Mohamad Kazemi et.al. Vehicle Recognition Based on Fourier, Wavelet and Curvelet Transforms - a Comparative Study; *IJCSNS VOL.7 No.2, February, 2007*.

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[11] Muhammad Azhar Iqbal et.al. Curvelet-based Image Compression with SPIHT; *IEEE Computer Society International Conference on Convergence Information Technology, 2007.*

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